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ELEMENTS

OF

PLANE GEOMETRY,

FOR

THE USE OF SCHOOLS.

BY N. TILLINGHAST.

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PREFACE.

This volume has been prepared with reference to what I believe to be the want of Schools ; it is short, because the time commonly devoted to Geometry in schools is very limited : several of the propositions usually contained in geometrical treatises are omitted, because they are considered too difficult for those for whom this work is designed ; and, for the same reason, rigor of demonstration has been departed from, in respect to some propositions which it was necessary to retain.

If the work has any merits, they will doubtless commend themselves to a candid public ; if not, it must meet the fate of many other things on which much labor has been bestowed in vain.

GEOMETRY.

BOOK I.

THE PRINCIPLES.—DEFINITIONS.

1. **Geometry** is that science which treats of Magnitudes. Magnitudes may be considered under three dimensions—length, breadth, thickness.

2. A *line* has length only, without breadth or thickness.

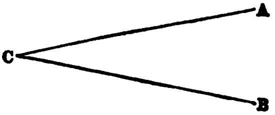
The extremities of a line are points ; hence,

3. A *point* has neither length, breadth, nor thickness, but position only.

4. A *straight line* is one, the direction of which is always the same.

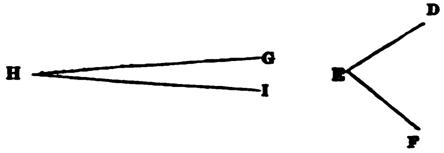
5. When two straight lines meet, the opening between them is called an *angle* ; the point of meeting is called the *vertex* ; and the lines themselves, which are said to *contain* the angle, are called the *sides*.

Thus the opening between the lines AC, CB, is called the angle made by those lines ; the point at which C is placed is called the vertex; and the lines AC, BC, are called the sides.

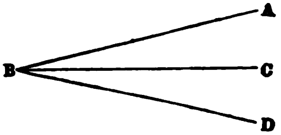


An angle is sometimes designated by the letter at the vertex; or, more frequently, it is designated by the three letters at the extremities of the sides, the letter at the vertex being always placed in the middle of the three letters; thus the angle ACB, denotes the angle having the vertex C, and contained by the sides AC, BC.

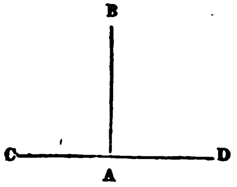
The pupil should be careful not to confound an angle with the length of its sides. It is evident, for example, that the lines DE, FE, make a much greater angle, that is, are much further apart, than the much longer lines GH, HI.



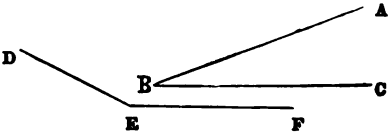
Angles, like all other quantities, are susceptible of being added to, and subtracted from each other: thus the angle ABD is the sum of the two angles ABC, CBD; and the angle ABC is the difference of the two angles ABD, CBD.



6. When a straight line, as AB, meets another straight line, as CD, so as to make the adjacent angles, CAB, BAD, equal to each other, each of these angles is called a *right angle*, and the line AB is said to be *perpendicular* to CD.



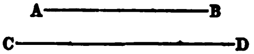
7. Every angle less than a right angle is called an



acute angle; and every angle greater than a right angle is called an *obtuse angle.* For example, ABC is an acute, and DEF is an obtuse angle.

8. A *plane* is a surface, in which, if *any* two points be taken, the straight line drawn between those points will lie wholly in the surface.

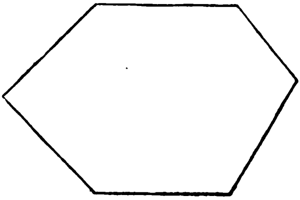
9. Straight lines are *parallel* when they have the same direction, as AB, CD.



Parallel lines cannot meet, how far so ever they are produced, (that is, continued.)

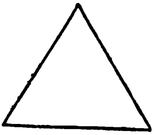
10. A *plane figure* is a plane terminated on all sides by lines.

11. If the bounding lines are straight, the figure is called a *polygon;* and the lines themselves, taken together, form the *contour*, or *perimeter* of the figure.

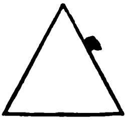


12. Among polygons, are more particularly distinguished the figure of three sides, called a *triangle*, and that of four sides, called a *quadrilateral.*

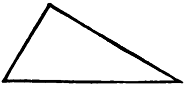
13. An *equilateral triangle* is one which has all its sides equal.



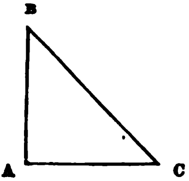
14. An *isosceles triangle* is one which has two equal sides.



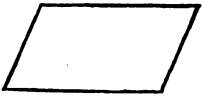
15. When no two sides are equal, the triangle is called *scalene*.



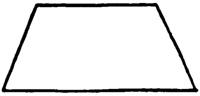
16. A *right-angled triangle* is one which has a righ angle. The side opposite the right angle is called the *hypotenuse*. Thus in the triangle ABC, right angled at A, BC is the hypotenuse.



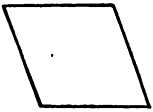
17. The *parallelogram* is a quadrilateral whose opposite sides are parallel.



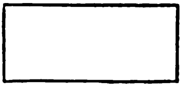
18. The *trapezoid* has only two of its opposite sides parallel.



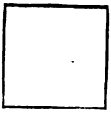
19. The *rhombus* is a parallelogram whose sides are all equal.

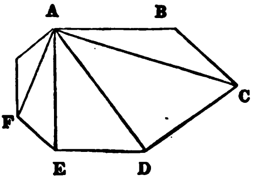


20. A *rectangle* is a parallelogram having all its angles right angles.



21. A *square* is a rectangle having all its sides equal.





22. A *diagonal* is a line which joins the vertices of two angles which are not adjacent to each other. Thus AC, AD, AE, AF, are diagonals.

23. Plane figures are *equal*, when, by supposing them to be applied to each other, they would coincide throughout; and they are said to be *equivalent*, when they enclose equal portions of space, and at the same time are incapable of such coincidence.

Definition of Terms employed.

An *axiom* is a self-evident proposition.

A *theorem* is a proposition which requires demonstration.

A *problem* proposes an operation to be performed.

A *corollary* is a consequence obviously resulting from a demonstration.

An *hypothesis* is a supposition.

A *scholium* is a remark subjoined to a demonstration.

Axioms.

1. Things which are equal to the same thing, are equal to each other.

2. Things which are double, triple, &c. of the same or equal things, are equal to each other.

3. Things which are the one half, one third, &c. of the same or equal things, are equal to each other.

4. If equals be added to, or taken from equals, the results will be equal.

5. If equals be added to, or taken from unequals, the results will be unequal.

6. The whole is greater than a part.

Explanation of Signs.

1. The sign $=$ signifies equality ; thus $A=B$ signifies that the quantity represented by A is equal to that represented by B ; and is read, A equal to B.

2. To represent that A is less than B, the expression $A<B$ is used, and is read, A less than B.

3. The sign $+$ is called *plus*, and signifies addition.

4. The sign $-$ is called *minus*, and signifies subtraction.

Thus, $A+B$ signifies the sum of the two quantities A and B, and is read, A plus B.

$A-B$ signifies the difference of these quantities, and is read, A minus B.

5. A dot . signifies that the quantities between which it is placed are multiplied together ; thus $A.B$ signifies that A is multiplied by B.

6. The expressions $A(B+C)$, $A(B+C+D)$, signify that, in the first, A is multiplied into the sum of the quantities B and C ; and in the second, that A is multiplied into the sum of the quantities B, C, and D.

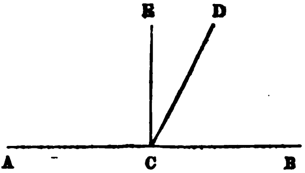
7. The figure 2 placed above and at the right hand of a quantity, signifies that the quantity is multiplied by itself; thus, A^2 represents that the quantity A is multiplied by itself ; it is read, A square.

PROPOSITION I. THEOREM.

When one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.

Let the straight line DC meet the line AB at C ;

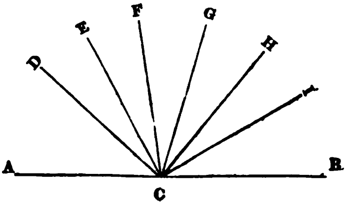
Fig. 1.



then we are required to prove that the sum of the two adjacent angles ACD, BCD, is equal to two right angles.

To do this, draw from the point C, where the two lines meet, the line CE perpendicular to AB; the angles ACE, ECB, will be right angles (Def. 6), and their sum, consequently, two right angles; but the opening between the lines AC and CD, which is the angle ACD, added to the opening between the lines DC and CB, which is the angle DCB, evidently makes the same amount of opening as that between the lines AC and CE, added to that between EC and CB, being in both cases the whole amount of opening between the parts AC and CB of the line AB; now the angle ACE+ECB we have seen is equal to two right angles, hence ACD+DCB must be equal to two right angles.

Fig. 2.



Corollary 1. The sum of all the angles, formed by any number of straight lines, drawn on the same side of another straight line, from any point in it, is equal to two right angles. For it is evident that the sum of all the openings between the lines drawn in the figure, and indeed between any number of lines drawn from the point C, on the same side of AB, is the same as the opening between any one of the lines, as CE, for example, and the parts AC and CB of the line AB; but the sum of these two angles, we learned above, is equal to two right angles; hence the sum of all the angles must also be equal to two right angles.

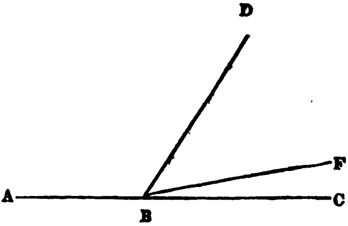
Cor. 2. The sum of all the angles formed on *both* sides the line A B, by lines meeting at C, is equal to four right angles, since those formed on each side are equal to two right angles.

Cor. 3. If one of the angles, formed by one straight line meeting another, is a right angle, the other angle must be also a right angle.

PROP. II. THEOREM.

If two straight lines meet, and, at the point of their meeting, a third be drawn so as to make with them adjacent angles, which, taken together, are equal to two right angles, the two first lines will form but one continued straight line.

Fig. 3.



Let the two straight lines, AB, CB, meet in the point B, and let a third line DB be drawn, so that the sum of the adjacent angles ABD, DBC, shall be equal to two right angles; then we have to prove that AB, BC, form one continued straight line, that is, that BC is the continuation of AB. If it is not, let BF be the continuation of AB, so that ABF shall form one continued straight line; then (by Prop. 1) the sum of the angles ABD, DBF, is equal to two right angles; but by the supposition ABD+DBC is equal to two right angles;

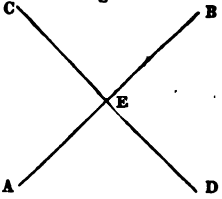
therefore ABD+DBF is equal to ABD+DBC ; now take away ABD from each of these equals, and DBF will be equal to DBC (Ax. 4) ; but DBF is only a part of DBC, and cannot be equal to it (Ax. 6), therefore BF cannot be a continuation of AB ; and as the same absurdity would arise from taking any line except BC as the continuation of AB, it follows that BC must be that continuation.

PROP. III. THEOREM.

If two straight lines intersect each other, the opposite angles, formed at their intersection, will be equal.

If the lines AB, CD, intersect at E, we have to prove that the opposite angles CEA, BED, are equal.

Fig. 4.



On account of the straight line CE meeting AB, the angles AEC+CEB are equal to two right angles (Prop. 1) ; and on account of the straight line BE meeting CD, the angles CEB +BED are equal to two right angles ; taking CEB from each of the equals, there remains CEA=BED (Ax. 4.) In a similar manner it may be shown that CEB=AED.

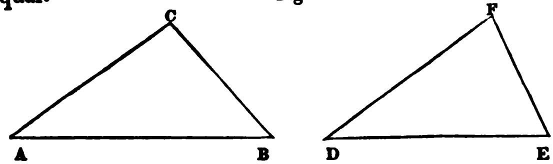
PROP. IV. THEOREM.

If two sides and the included angle, in one triangle, be respectively equal to the two sides and included angle in another triangle, these two triangles will be equal. (Def. 23.)

2

In the two triangles ABC, DEF, let the sides AC, AB, and the included angle at A, be respectively equal to the sides DF, DE, and the included angle at D; then we have to prove that the two triangles are equal.

Fig. 5.



Suppose the triangle DEF to be placed upon the triangle ABC, so that DE lies upon AB, the point D at A, the point E at B; now, since the angle FDE is equal to the angle CAB, the line DF will take the direction AC, and, being equal to AC in length, its extremity F will fall on C, and consequently the line FE will coincide with CB, the angle at F with the angle at C, and the angle at E with the angle at B.

PROP. V. THEOREM.

If two angles and an included side in one triangle be equal respectively to two angles and an included side in another, the triangles will be equal.

Let, (as in the figure above,) the angle A be equal to the angle D, the angle B equal to the angle E, and the side AB equal to DE; then we have to prove that the triangles ABC, DEF, are equal.

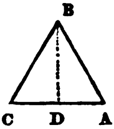
Suppose the triangle DEF to be placed upon ABC, so that DE shall coincide with AB; now, since the angle FDE is equal to the angle CAB, DF will take the direction AC; and, since the angle DEF is equal to the angle ABC, EF will take the direction BC; hence

DF, EF, will intersect at C, and DF be equal to AC, EF equal to BC, and the angle at F equal to the angle at C.

PROP. VI. THEOREM.

In an isosceles triangle, the angles opposite the equal sides are equal.

Fig. 6.



Let AB, BC, be the equal sides ; then we have to prove that the angle A is equal to C.

Draw from B the line BD, so as to divide the angle B into two equal parts ; then in the two triangles ABD, CBD, we have the two sides AB, BD, and the included angle ABD, in the one, respectively equal to the two sides BC, BD, and the included angle CBD, in the other ; hence the two triangles are equal (Prop. 4), and the angle A is equal to the angle C.

Cor. 1. From the equality of the two triangles ABD, CBD, we see further that AD is equal to DC, and the angles BDA, BDC, are equal, and hence (Def. 6) right angles.

Cor. 2. Every equilateral triangle is therefore equiangular.

Scholium. In a triangle any side may be assumed as the *base*, and then the *vertex* is the vertex of the opposite angle.

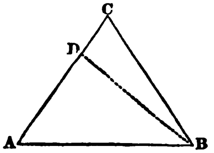
In an isosceles triangle, however, we generally assume as the base the side which is not equal to either of the other two.

PROP. VII. THEOREM.

(Converse of Prop. VI.)

If two angles of a triangle are equal, the sides opposite them are also equal, and the triangle is isosceles.

Fig. 7.



In the triangle ABC, let the angles ABC, BAC, be equal, then we have to prove that the sides AC, BC, will be also equal.

If they are not equal, one of them must be the longer; let us suppose it to be AC, and make AD=CB, and draw the line BD; then we have the side AD of the triangle ABD equal to the side BC of the triangle ACB, the side AB common to both triangles, and the angle DAB of one equal to the angle CBA of the other (by hypothesis), hence the two triangles ABD, ABC, will be equal (Prop. 4); but this is manifestly impossible, since the triangle ABD is only a part of ABC; therefore the line AC cannot be longer than CB.

In a similar manner it may be proved that BC cannot be longer than AC; therefore they must be equal.

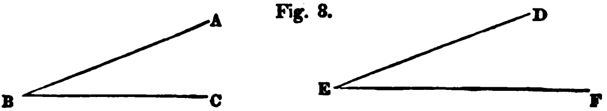
Cor. Every equiangular triangle is equilateral.

PROP. VIII. THEOREM.

Two angles are equal when they have their sides parallel and directed the same way.

Let AB be parallel to DE, and CB parallel to FE; then we have to prove that the angle ABC is equal to the angle DEF.

Fig. 8.

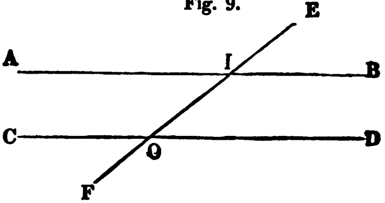


The lines BA, ED, have the same direction, as have also BC, EF (Def. 9); therefore, if we suppose DEF applied to ABC, so that the point E shall fall on B, and EF lie on BC, the line ED will pursue the same direction as BA; and hence the angle DEF is equal to ABC.

PROP. IX. THEOREM.

If two parallel lines are cut by a third straight line, the alternate angles will be equal.

Fig. 9.



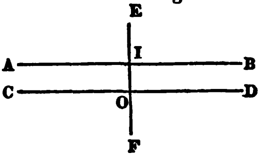
Let the parallels AB, CD, be cut by the line EF; then we have to prove that the alternate angles AIF, EOD, are equal.

The angles EIB, EOD, are equal by the last proposition; but EIB is equal to AIF (Prop. 3); hence EOD must be equal to AIF (Ax. 1).

Cor. 1. We see by the demonstration that EIB, EOD, are equal; these are called *opposite exterior and interior angles.*

Cor. 2. A line which is perpendicular to one of two parallels will be perpendicular to the other. Let EF be drawn perpendicular to AB; now, because it cuts the two parallels, it makes the alternate angles AIF,

Fig. 10.



EOD, equal, and since AIO is a right angle, EOD must be one also.

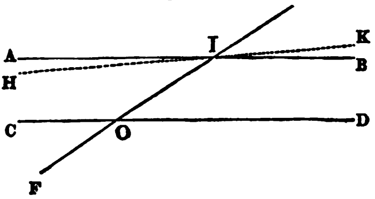
PROP. X. THEOREM.

(Converse of Prop. IX.)

If two straight lines are cut by a third, making the alternate angles equal, the two first lines are parallel.

Let the angles AIF, EOD, be equal; then we have to prove that AB, CD, are parallel. If they are not parallel, draw through I, the line HK, parallel to CD; then will HIF be equal to EOD (Prop. 9); but, by the hypothesis, EOD is equal to AIF; hence HIO must be equal to AIF, which is manifestly impossible, since HIF is only a part of AIF; therefore HK cannot be parallel to CD; and, as the same can be proved with regard to any other line than AB, it follows that AB is parallel to CD.

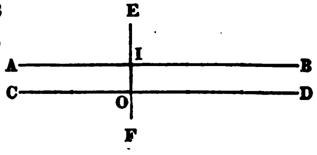
Fig. 11.



Cor. 1. When the exterior and interior angles EIB, EOD, are equal, AB, CD, must be parallel; because when those angles are equal the alternate angles are equal also.

Cor. 2. Two straight lines which are perpendicular to a third line are parallel to each other; because the alternate angles AIF, EOD, being right angles, are equal to each other.

Fig. 12.

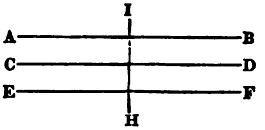


Cor. 3. Hence, if to each of two parallels, perpendiculars be drawn, these perpendiculars will be parallel, for each will be perpendicular to both parallels (Prop. 9, Cor. 2).

PROP. XI. THEOREM.

If two straight lines be parallel to a third, they will be parallel to each other.

Fig. 13.



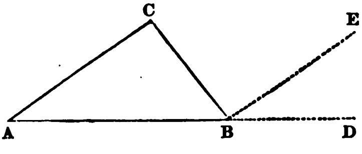
Let CD, EF, be parallel to AB; then we have to prove that they will be parallel to each other.

Draw IH perpendicular to AB, it will also (by Prop. 9, Cor. 2) be perpendicular to both CD and EF; hence these lines, being both perpendicular to the same line, are parallel to each other (Prop. 10, Cor. 2).

PROP. XII. THEOREM.

If one side of a triangle be produced, the exterior angle, so formed, will be equal to the sum of the two interior opposite angles.

Fig. 14.



Let the side AB, of the triangle ABC, be produced towards D; then we have to prove that the exterior angle CBD, thus formed, will be equal to the sum of the interior and opposite angles BAC, BCA.

From the point B draw the line BE parallel to AC; then will the angles EBD, BAC, be equal (Prop. 9, Cor. 1), and the angles CBE, BCA, be equal (Prop.

9); hence CBD, which is the sum of the two angles CBE, EBD, is equal to the sum of the two angles BAC, BCA.

Cor. 1. The sum of the three angles of a triangle is equal to two right angles; for CBD+CBA is equal to two right angles (Prop. 1); now we have just seen that CBD is equal to BAC+BCA; hence BAC+BCA+CBA is equal to two right angles.

Cor. 2. Hence, if two angles in one triangle be equal to two angles in another triangle, the third angle in the one will be equal to the third angle in the other.

Cor. 3. In any triangle there can be but one right angle; for if there could be two, the third angle would be nothing.

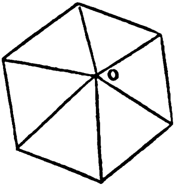
Cor. 4. In a right-angled triangle the sum of the two acute angles is equal to one right angle.

PROP. XIII. THEOREM.

In any polygon, the sum of all the angles is equal to as many times two right angles as the figure has sides, less four right angles.

From any point O within the polygon, draw lines to the vertices of all the angles. There will evidently be formed as many triangles as the figure has sides, and the sum of all the angles of the polygon will be less than the sum of all the angles of the triangles, by the angles formed at the point O. Now the sum of all the angles of *each* triangle is equal to two right angles (Prop. 12); and the sum of all the angles

Fig. 15.



formed about O is four right angles (Prop. 1, Cor. 3); hence the sum of all the angles of the polygon is equal to as many times two right angles as the figure has sides, less four right angles.

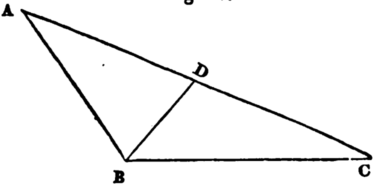
Cor. For example, the sum of all the angles of a six-sided figure is eight right angles; since it is equal to as many times two right angles as the figure has sides, (or twelve right angles,) less four. If all the six angles are equal to each other, each of them will be the sixth part of eight right angles, or $\frac{8}{6}=\frac{4}{3}$ of a right angle.

PROP. XIV. THEOREM.

In a triangle, the angle which is opposite the longest side is the largest.

In the triangle ABC, let the side AC be longer than AB; then we have to prove that the angle ABC is greater than the angle C. On AC take AD=AB, and draw BD. Now, because the sides AB, AD, of the triangle ABD, are equal, the angles ABD, ADB, must be equal (Prop. 6); but the angle ADB, being an exterior angle of the triangle DBC, is equal to the sum of the angles DBC, DCB, (Prop. 12), and is therefore larger than C; hence ABD must be larger than C; and ABC, which exceeds ABD, must be still larger than C. In a similar manner, if the side AC is taken longer than BC, may we prove the angle ABC to be larger than A.

Fig. 16.

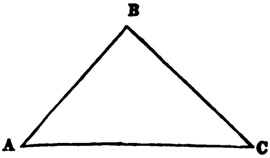


PROP. XV. THEOREM.

(Converse of Prop. XIV.)

In a triangle, the side which is opposite the largest angle is the longest.

Fig. 17.



In the triangle ABC, take the angle B larger than the angle A; then we have to prove that the side AC is longer than BC.

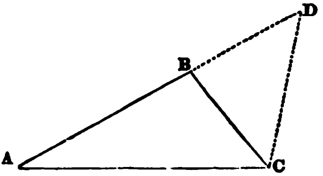
If AC is not longer than BC, it is either equal to it, or less than it; if it were equal to it, the angle B would be equal to the angle A (Prop. 6); and if it were less than it, the angle A would be larger than B (Prop. 14); both of these results are contrary to the supposition; hence AC must be longer than BC.

Cor. Hence in a right-angled triangle the hypotenuse is the longest side. (Prop. 12, Cor. 3.)

PROP. XVI. THEOREM.

The sum of any two sides of a triangle is longer than the third side.

Fig. 18.



In the triangle ABC, we have to prove, for example, that AB+BC is longer than AC.

Produce AB until BD is equal to BC, and draw CD; then the angle BCD=D (Prop. 6); therefore ACD, which is larger than BCD, is also larger than D; hence AD is longer than AC (Prop. 15); but

AD is equal to AB+BC ; hence the sum of the two sides AB, BC, is longer than the third side AC ; the same thing may be proved of the sum of any other two sides.

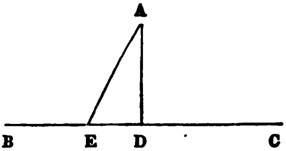
PROP. XVII. THEOREM.

A perpendicular is the shortest distance from a point to a straight line.

Let A be the point, BC the straight line ; we have to prove that the perpendicular AD is shorter than any other line which can be drawn from A to BC.

Draw, for example, AE ; then is AD$<$AE (by Prop. 15, Cor.)

Fig. 19.

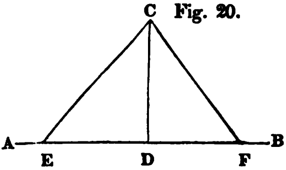


PROP. XVIII. THEOREM.

If a perpendicular be drawn to a given straight line, and, from any point on the perpendicular, oblique lines be drawn to the given line, so as to cut off equal distances from the foot of the perpendicular, these oblique lines will be equal.

Let AB be the given straight line, CD the perpendicular, and DE$=$DF ; then we have to prove that the oblique lines CE, CF, drawn from any point C of the perpendicular, are equal.

Fig. 20.



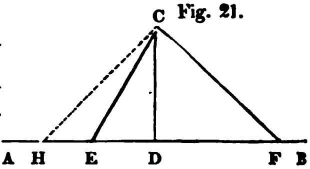
In the two right-angled triangles EDC, FDC, we have the sides ED, DC, and the included angle EDC, of the one, respectively equal to the sides DF, DC, and the included angle FDC, of the other; hence these triangles are equal (Prop. 4), and CE=CF.

PROP. XIX. THEOREM.

If a perpendicular be drawn to a given straight line, and, from any point on the perpendicular, oblique lines be drawn to the given line, those which are furthest from the perpendicular will be longest.

Let CD be perpendicular to the given line AB, and DE<DF; then we have to prove that the oblique line CF is longer than CE.

Fig. 21.



Make DH=DF and draw CH; then will CH be equal to CF (Prop. 18), and the angle CHF=CFH (Prop. 6); but the angle CEF, being the exterior angle of the triangle CHE, is larger than CHF (Prop. 12), and therefore larger than CFH; hence, in the triangle CEF, the side CF is longer than CE (Prop. 15).

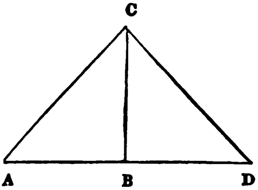
Cor. 1. Hence from the same point to the same straight line only two *equal* oblique lines can be drawn; for if three could be drawn, two of them must be on the same side of the perpendicular, which is impossible (by the Prop.)

Cor. 2. If the oblique lines are equal, the distances cut off from the foot of the perpendicular must be equal; since we see from the proposition, that if these distances are unequal, the oblique lines will be also unequal.

PROP. XX. THEOREM.

Two right-angled triangles, having the hypotenuse and one side of one, respectively equal to the hypotenuse and one side of the other, are equal.

Fig. 22.



Let the equal sides be so placed as to coincide (as in the figure); then in the right-angled triangles ACB, DCB, we have the hypotenuse AC, and side CB, of one, respectively equal to the hypotenuse CD, and side CB, of the other; and are required to prove that the two triangles are equal.

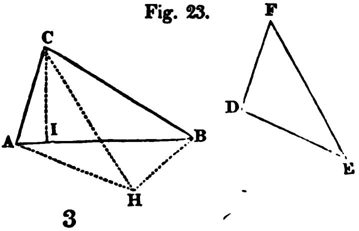
Because AC and CD are equal, AB and BD must be equal (Prop. 19, Cor. 2); hence the two triangles ACB, DCB, having two sides and included angle of one, equal to two sides and included angle of the other, are equal (Prop. 4).

PROP. XXI. THEOREM.

If two sides of one triangle be respectively equal to the two sides of another, but include a greater angle, then will the third side of the former exceed the third side of the latter.

Let ABC, DEF, be two triangles, having the sides

Fig. 23.



AC, CB, of the one, equal respectively to the sides DF, FE, of the other, and the included angle ACB, of the first, greater than the included angle DFE of the second; then we have to prove that the third side, AB, is longer than DE.

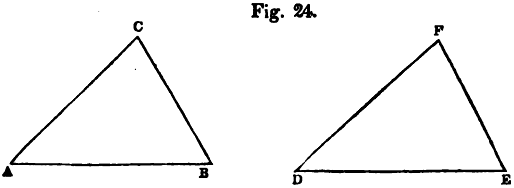
If the triangles are not isosceles, let CB be taken as the longer side; draw the perpendicular CI, and then the line CH, so as to make the angle ACH=DFE; make CH=FE=CB; draw AH, HB; then, since CB is longer than AC, and CH is made equal to CB, the point H must fall outside of the line AB (Prop. 19). Now the two triangles ACH, DFE, are equal, because they have the two sides AC, CH, and the included angle ACH of the one, equal respectively to the two sides DF, FE, and the included angle DFE of the other; hence the line AH is equal to DE (Prop. 4). Again, the angle CBH=CHB, (because the side CH was made equal to CB,) (Prop. 6); but the angle ABH is less than CBH, and the angle AHB is greater than CHB; hence AHB is greater than ABH; therefore in the triangle AHB, the side AB is longer than AH (Prop. 15), and consequently longer than DE.

PROP. XXII. THEOREM.

Two triangles, having the three sides of the one respectively equal to the three sides of the other, are equal.

In the triangles ABC, DEF, let AC=DF, CB

Fig. 24.



=FE, AB=DE ; then we have to prove the triangles to be equal.

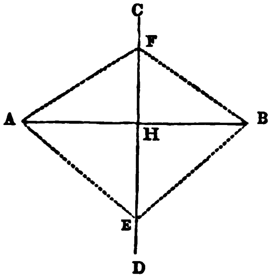
If the angle C is not equal to the angle F, it must be either larger or smaller than it ; but it cannot be larger, for then AB would be larger than DE ; nor can it be smaller, for then DE would be larger than AB (Prop. 21) ; both of these results are contrary to the supposition, therefore C=F ; and hence the triangles, having two sides and included angle of one, equal to two sides and included angle of the other, each to each are equal (Prop. 4).

Scholium. The equal angles are opposite the equal sides.

PROP. XXIII. THEOREM.

If a straight line be so situated that of any two points in it each is equally distant from the extremities of another straight line, it will bisect the second line and be perpendicular to it.

Fig. 25.



Let us suppose each of the points E, F on the line CD, to be situated at equal distances from the extremities of the line AB, so that AF=FB, and AE=EB ; then we have to prove that AH=HB, and that the angle AHF=BHF (Def. 6).

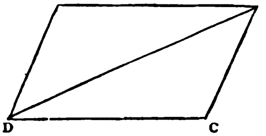
In the triangles AEF, BEF, we have AF=FB, AE=BE, and FE common ; hence these triangles are equal (Prop. 22), and the angle AFH=BFH ; now in the triangles AFH, BFH, we have the sides AF, FH, and the included angle AFH of the one, equal respectively to the sides BF, FH, and the included angle BFH of the other; hence these triangles are equal (Prop. 4), and AH=HB, and the angle AHF=BHF.

PROP. XXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal.

Having the parallelogram ABCD (Def. 17), we have to prove that AB=DC, AD=BC, A=C, B=D.

Fig. 26.



Draw a diagonal DB. Now because AB, DC, are parallel, the angles ABD, BDC, are equal, and because AD, BC, are parallel, the angles ADB, DBC, are equal (Prop. 9); then the triangles ABD, CDB, having the two angles ABD, ADB, and the included side DB of the one, equal respectively to the angles, BDC, DBC and the included side BD of the other, are equal (Prop. 5), AB=DC, AD=BC, the angle A=C, and the angle ADC, composed of the two angles ADB, BDC, equal to ABC, composed of CBD, ABD.

Cor. A diagonal divides a parallelogram into two equal triangles.

PROP. XXV. THEOREM.

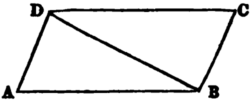
(Converse of Prop. XXIV.)

If the opposite sides of a quadrilateral be equal, they are also parallel, and the figure is a parallelogram.

Let AB=DC, AD=BC; then we have to prove that AB is parallel to DC, and AD is parallel to BC.

Draw the diagonal DB; then in the triangles ABD, BCD, we have AB=DC, AD=BC, and BD common; hence these triangles are equal (Prop. 22), and the angles ABD, ADB, of the one, respectively equal to the angles BDC, DBC, of the other; and, consequently, AB is parallel to DC, and AD to BC (Prop. 10), and the figure is a parallelogram.

Fig. 27.



PROP. XXVI. THEOREM.

If two of the opposite sides of a quadrilateral are equal and parallel, the other two sides will be equal and parallel, and the figure will be a parallelogram.

In the quadrilateral ABCD (preceding diagram), let AB be equal and parallel to DC; then we have to prove that AD is equal and parallel to BC.

The line DB cutting the parallels AB, DC, makes the alternate angles ABD, BDC, equal (Prop. 9); hence the triangles ABD, CDB, having the two sides AB, BD, and the included angle ABD of the one, equal respectively to the sides CD, BD, and the included angle BDC of the other, are equal (Prop. 4), AD=BC, the angles ADB, CBD, equal, and consequently AD parallel to BC (Prop. 10); hence the figure is a parallelogram.

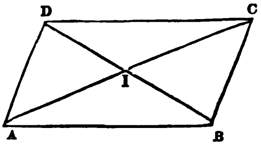
PROP. XXVII. THEOREM.

The diagonals of a parallelogram mutually bisect each other.

We have to prove that AI=IC, and DI=IB.

The angles ACD, CAB, are equal, and the angles

Fig. 28.



BDC, DBA, are also equal (Prop. 9); hence the triangles DIC, BIA, having the angles ICD, IDC, and the included side DC of the one, equal respectively to the angles IAB, IBA, and the included side AB of the other, are equal, and AI=IC, DI=IB.

BOOK II.

DEFINITIONS.

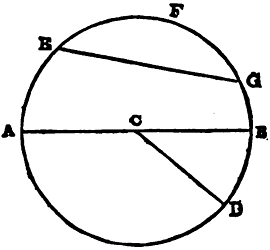
1. Every line which is not straight, is called a *curve* line.

2. A *circle* is a space enclosed by a curve line, every point of which is equally distant from a point within; which point is called the *centre*.

3. The boundary of a circle is called its *circumference*.

4. A *radius* is a line drawn from the centre to the circumference. All radii are equal.

5. A *diameter* is a line which passes through the centre, and has its extremities in the circumference. A diameter therefore is double the radius.



In the circle enclosed by the circumference AEFBD, C is the centre, CD is a radius, and AB is a diameter.

6. An *arc* is any portion of the circumference, as EFG.

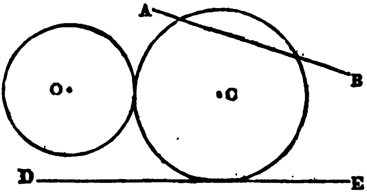
7. The *chord* of an arc is a straight line joining its extremities. EG is the chord of the arc EFG.

8. A *segment* of a circle is the portion inclu between an arc and its chord. The space EF

between the arc EFG and the chord EG, is a segment.

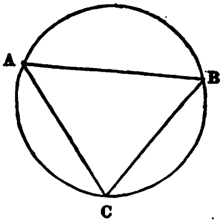
9. A *sector* of a circle is the portion included between two radii. BDC, the part of the circle between BC, CD, and the arc BD, is a sector.

10. A *secant* is a line cutting the circumference of the circle, and lying partly within and partly without the circle.



11. A *tangent* is a line which touches the circumference in only one point, which is called the *point of contact*. AB is a secant, and DE a tangent to the circle whose centre is C.

12. One circle touches another when their circumferences have only one point in common.



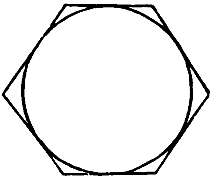
13. A line is *inscribed* in a circle when its extremities are in the circumference; as the line AB.

14. An angle is *inscribed*, when its vertex is in the circumference; as the angle A.

15. A triangle is *inscribed*, when the vertices of its three angles are in the circumference; as the triangle ABC.

16. Generally any polygon is inscribed, when the vertices of all its angles are in the circumference; the circle is then said to *circumscribe* the polygon.

17. A circle is inscribed in a polygon, when its circumference touches each side of the polygon; and the polygon is then said to *circumscribe* the circle.



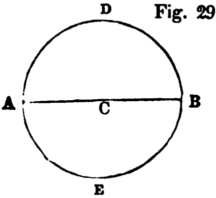
18. *Equal circles* are those which are described with equal radii.

PROP. I. THEOREM.

Every diameter divides the circle and its circumference into two equal parts.

To prove this, suppose the portion ADB applied upon the semi-circle AEB; the curve line ADB must fall exactly upon the curve line AEB, because otherwise there would be, in one or the other, points unequally distant from the centre C, which is impossible (Def. 2); hence the circle and its circumference are divided into equal parts by the diameter AB.

Fig. 29.



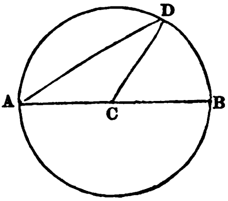
Cor. Hence in equal circles, the semi-circles and semi-circumferences are respectively equal.

PROP. II. THEOREM.

Every chord is less than the diameter.

We have to prove that the chord AD is less than the diameter AB.

Fig. 30.

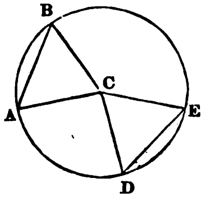


Draw CD. Now in the triangle ADC, AD$<$AC$+$CD (Book I. Prop. 16); but CD$=$CB (Def. 4); hence AD$<$AC$+$CB; that is, AD$<$AB.

PROP. III. THEOREM.

In the same, or in equal circles, equal angles at the centre are subtended by equal arcs.

Fig. 31.



Let C be the centre of the circle, and let the angle ACB be equal to ECD; then we have to prove that the arcs AB, DE, which subtend these angles, are equal.

Draw the chords AB, DE; then since radii are all equal (Def. 4), the triangles ACB, DCE, have two sides and the included angle of the one, equal respectively to two sides and the included angle of the other, hence they are equal; so that if ACB be applied to DCE, there will be an exact coincidence (B. I. Def. 23), the point A would fall on D, and the point B on E; the two extremities therefore of the arc AB thus coinciding with those of DE, all the intermediate parts must coincide, inasmuch as they are equally distant from the centre (Def. 2).

Cor. 1. It follows moreover that equal angles at the centre are subtended by equal chords.

Cor. 2. If the angle at the centre be bisected, both the arc and chord subtending it are also bisected.

Cor. 3. If the angles are unequal the chords will be unequal.

Scholium. The above reasoning applies obviously to the case of *equal* circles, as one would entirely coincide with the other.

PROP. IV. THEOREM.

(Converse of Prop. III.)

In the same circle, or in equal circles, equal arcs subtend equal angles at the centre.

Let (as in Prop. 3) the arc AB be equal to the arc DE; then we have to prove that the angles ACB, DCE, are equal.

If ACB were applied to DCE, the arcs AB, DE, being equal, would coincide, the point A would fall upon D, and B upon E, the line AB would coincide with DE; hence the two triangles ACB, DCE, having the three sides of the one respectively equal to the three sides of the other, are equal (B. I. Prop. 22), and the angles ACB, DCE, are equal.

Cor. 1. Equal chords subtend equal angles at the centre.

Cor. 2. Therefore equal chords subtend equal arcs, and *conversely* equal arcs are subtended by equal chords.

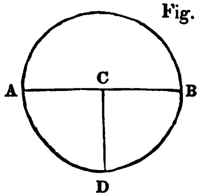
Cor. 3. If the chords are unequal the angles and arcs which they subtend are unequal.

PROP. V. THEOREM.

An angle at the centre, subtended by half a semi-circumference, is a right angle.

Let the point D be the middle of the semi-circumference ADB; then we have to prove that the angle ACD, which is subtended by half the semi-circumference, is a right angle.

Fig. 32.

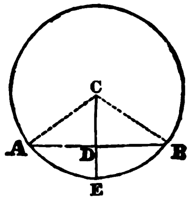


The arcs AD, DB, are equal, being each half of a semi-circumference; therefore the angles ACD, DCB, are equal (Prop. 4); but these angles are adjacent; hence each is a right angle. (B. I. Def. 6.)

PROP. VI. THEOREM.

A line drawn from the centre of a circle perpendicular to a chord, bisects it, and also the arc which it subtends.

Fig. 33.



Let the line CE be drawn from C, the centre, perpendicular to AB; then we have to prove that it will bisect AB, and also the arc AEB.

Draw the radii AC, BC; these are equal (Def. 4), therefore AD=DB (B. I. Prop. 19, Cor. 2); hence the two triangles ACD, BCD, having the three sides of the one equal to the three sides of the other, are equal, (B. I. Prop. 22), and the angles ACD, BCD, are equal; and therefore the arcs AE, EB, are equal (Prop. 3, Cor. 2).

Cor. 1. Hence the line joining the centre of the circle and the middle of the chord, or the middle of the arc, is perpendicular to the chord.

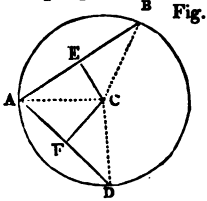
Cor. 2. A perpendicular through the middle of the chord passes through the centre, and through the middle of the arc, and bisects the angle at the centre which the chord subtends.

PROP. VII. THEOREM.

Through three given points, not in the same line, one circumference may be made to pass, and only one.

To prove this, let A, B, D, be the given points, and join them by the lines AB, AD ; from E, F, the middle points of these lines, erect the perpendiculars EC, FC, which will meet in some point, C ; for if they do not meet, they must be parallel, and then the lines AB, AD, which are perpendicular to them, would also be parallel (B. I. Prop. 10, Cor. 3), or else form but one straight line ; but they meet at A, and are therefore not parallel ; and by the hypothesis BAD is not a straight line : hence EC, FC, will meet ; and if from the point C, in which they meet, as a centre, we describe a circumference, with the radius CA, it will pass through the three points A, B, D. For draw CA, CB, CD, and because AC, BC, meet AB at equal distances from the foot of the perpendicular CE, they are equal (B. I. Prop. 18) ; for the same reason AC, DC, are equal ; hence the three points A, B, D, lying at equal distances from C, are in the circumference of a circle whose centre is C, and radius CA.

Fig. 34.



Again, only one circumference can pass through the three points A, B, D, because the lines EC, FC, intersect each other but once, and therefore there can be but one centre, and with the same centre and the same radius, it is obvious only one circumference can be described.

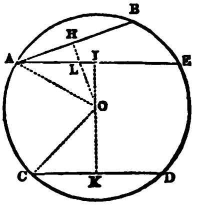
Cor. One circumference cannot cut another in more than two points.

PROP. VIII. THEOREM.

Two equal chords are equally distant from the centre.

Of two unequal chords the shorter is at the greater distance from the centre.

Fig. 35.



1st. Let AB, CD, be equal, then we have to prove that their distances from the centre, which are measured by the perpendicular lines OH, OK, are equal.

The lines OH, OK, being drawn from the centre perpendicular to the chords AB, CD, bisect them (Prop. 6); therefore AH, the half of AB, is equal to CK, the half of CD (B. I. Ax. 3). Now in the right-angled triangles AOH, CKO, AO is equal to OC, and AH to CK; hence these triangles are equal (B. I. Prop. 20), and $OH=OK$.

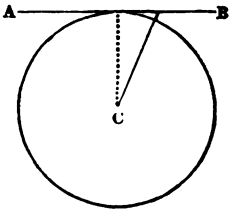
2d. Let AE be longer than AB; we have to prove that AB is further from the centre than AE. Draw OI perpendicular to AE. OL is longer than OI (B. I. Prop 15, Cor.), but $OL<OH$; hence $OI<OH$; but OI, OH, measure the distances from O to AE, AB.

PROP. IX. THEOREM.

Every straight line perpendicular to a radius at its extremity is a tangent to the circle.

Conversely. Every tangent to a circle is perpendicular to the radius drawn to the point of contact.

Fig. 36.



1st. Let AB be perpendicular to the radius CD, at its extremity D; then we have to prove that AB is a tangent to the circle. To do this we have only to show that AB cannot touch the circle in any other point than D (Def. 11). Draw any other line CE, from the centre to AB; now CE is longer than CD (B. I. Prop. 15, Cor.), hence the point E must be without the circumference, and as the same can be proved for any point on the line AB except D, it follows that AB touches the circumference in but one point, and is therefore tangent to the circle.

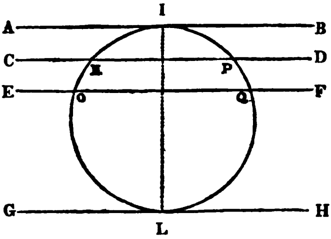
2d. Conversely. If AB is a tangent, that is, touches the circumference in but one point, we have to prove that CD, drawn from the centre to the point of contact, is perpendicular to AB. If it were not perpendicular it would not be the shortest distance from C to AB (B. I. Prop. 17), and consequently AB would pass inside of the circumference; but AB does not pass inside of the circumference (Def. 11); therefore CD is perpendicular to it.

PROP. X. THEOREM.

The arcs of a circle intercepted by two parallels are equal.

1st. Let the parallels be tangents, as AB, GH, then the intercepted arcs INOL, IPQL, being semi-circumferences, are equal (Prop. 1).

Fig. 37.



2d. If one is a tangent AB, and the other a secant CD; then will the arcs IN, IP, be equal (Prop. 6).

3d. If both are secants CD, EF; then OI=IQ, NI=IP (Prop. 6); consequently NO, the difference between OI and NI, is equal to PQ, the difference between IQ and IP (B. I. Ax. 4).

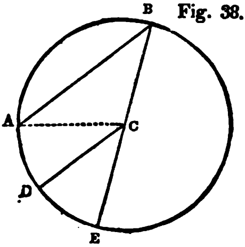
PROP. XI. THEOREM.

An inscribed angle is equal to an angle at the centre, whose sides include half the arc included between the sides of the inscribed angle.

There will be three cases; *first*, when the centre of the circle is *on* one of the sides of the inscribed angle; *second*, when the centre is *within* the angle; *third*, when the centre is *without* the angle.

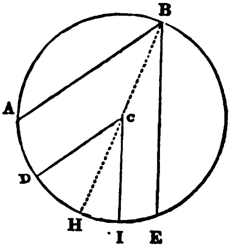
1*st*. Let ABE be the inscribed angle, the centre of the circle being in the side BE. Draw CD to the middle of the arc AE; then we have to prove that ABE=DCE.

Fig. 38.



Draw AC. Now the triangle ACB being isosceles, the angles ABC, BAC, are equal (B. I. Prop. 6); and the angle ACE, exterior to the triangle ACB, is equal to the sum of ABC and BAC (B. I. Prop. 12), or, since they are equal, to the double of either of them; hence the angle ACE is double the angle ABE; but ACE is double DCE, because the arc AE is bisected at D (Prop. 4), hence ABE=DCE.

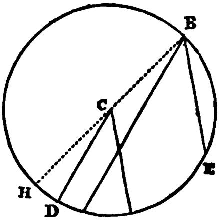
Fig. 39.



2*dly*, when the centre is *within* the angle. Let ABE be the inscribed angle. Draw the diameter BH; CD to the middle of the arc AH, and CI to the middle of the arc HE; then the arc DI will be half of the arc AE; and we have to prove that ABE=DCI.

By the preceding case ABH=DCH, and EBH=ICH; hence ABH+EBH=ABE, is equal to DCH+ICH=DCI (B. I. Ax. 4).

Fig. 40.



3dly, when the centre is *without* the angle. Let ABE be the inscribed angle. Draw BH the diameter, CI to the middle of the arc HE, and CD to the middle of the arc HA. The arc AE is the difference of the two arcs HE, HA; and the arc DI is the difference of the arcs HI, HD; but HI is half of HE, and HD is half of HA; hence DI is half of AE; and we have to prove that the angles ABE, DCI, are equal. By the first case, HBE =HCI, and HBA=HCD; hence the difference of HBE and HBA, which is ABE, is equal to the difference of HCI and HCD, which is DCI.

Cor. 1. Hence an angle at the centre is double one at the circumference subtending the same arc.

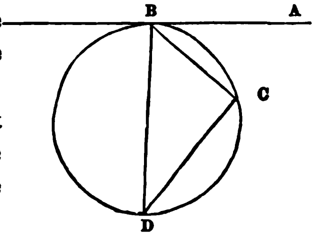
Cor. 2. Inscribed angles subtended by the same or equal arcs are equal, because they are each equal to the same angle at the centre.

Cor. 3. An inscribed angle, which is subtended by a semi-circumference, is a right angle; because it is equal to an angle at the centre subtended by half a semi-circumference, and this is equal to a right angle (Prop. 5).

PROP. XII. THEOREM.

An angle formed by a tangent and chord, is equal to an inscribed angle whose sides enclose the same arc that is subtended by the given chord.

Fig. 41.



Let AB be the tangent, BC the chord. Draw the diameter BD ; then we have to prove that the angle ABC, formed between the tangent and chord, is equal to the inscribed angle CDB, whose sides enclose the arc BC.

The angles ABC, CBD, taken together compose the angle DBA, which is a right angle (Prop. 9, 2d part) ; but in the triangle BCD, the angle C is a right angle (Prop. 11, Cor. 3), and consequently the angles DBC, CDB, taken together make a right angle (B. I. Prop. 12, Cor. 4); hence the sum of the angles ABC, CBD, is equal to the sum of the angles CDB, CBD (since each sum is equal to a right angle); and taking CBD from each of these equals, we shall have ABC=CDB (B. I. Ax. 4).

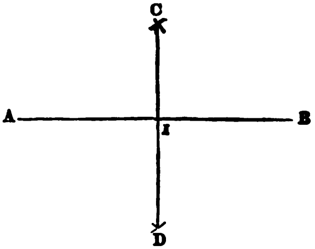
BOOK III.

PROBLEMS.

PROP. I. PROBLEM.

To divide a given straight line into two equal parts.

Fig. 42.

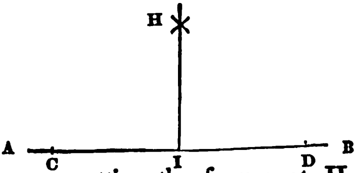


Let AB be the given line. From A as a centre, with a radius longer than half of AB, describe an arc above the line AB, and another below it; then, with B as a centre and the same radius, describe two arcs intersecting the two first at C and D, and draw the line C D, it will bisect AB at I: for, the points C, D, being equally distant from the extremities A, B, of the line AB, CD, bisects AB (B. I. Prop. 23).

PROP. II. PROBLEM.

At a given point on a straight line to erect a perpendicular to the line.

Fig. 43.



Let I be the given point, on the straight line AB. From I lay off the equal distances IC, ID. With C as a centre, and a radius longer than IC, describe an arc, and from D as a centre, with the same radius, describe an arc cutting the former at H.

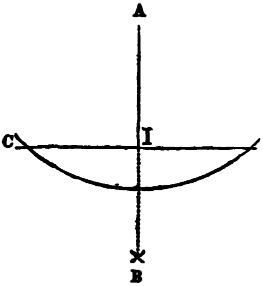
Draw HI, and it will be the perpendicular required, cause the line HI has two of its points, each of whicl equally distant from C and D, the extremities of C: hence IH is perpendicular to CD (B. I. Prop. 23).

PROP. III. PROBLEM.

From a given point without a straight line, to dra perpendicular to the line.

From A, the given point, as a centre, with a radius sufficiently long, describe an arc cutting the line in C, D; and from these points as centres, with a radius greater than half CD, describe two arcs cutting each other at B. Draw from B to A a line, and AI will be the perpendicular sought; because it has two points A, B, ea equally distant from C and D (B. I. Prop. 23).

Fig. 44.

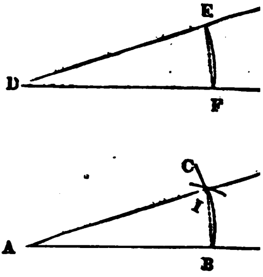


PROP. IV. PROBLEM.

At a given point in a straight line, to make an an equal to a given angle.

Let A be the given point, on the line AB, and let D be the given angle. From D as a centre, with any radius, describe the arc EF, and from A as a centre, with the same radius, describe an arc CB; then, with B as a centre, and a radius equal to

Fig. 45.



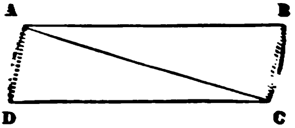
the chord FE, describe an arc cutting the arc CB in I; draw AI, and the angle BAI will be equal to the given angle D; because the angles D and A, having equal radii and equal chords, are equal (B. II. Prop. 4, Cor. 1).

PROP. V. PROBLEM.

Through a given point to draw a straight line parallel to a given straight line.

Let A be the given point, DC the given line. From A as a centre, with any radius longer than the shortest distance from A to DC, describe the arc BC; from C as a centre, with the radius CA, describe the arc AD; draw the chord AD, and with C as a centre and AD as radius, describe an arc cutting the arc BC in B. Through A and B, draw the line AB, and it will be the required parallel. For the alternate angles ACD, CAB, having equal radii and equal chords, are equal (B. II. Prop. 4, Cor. 1); and hence AB, DC, are parallel (B. I. Prop. 10).

Fig. 46.

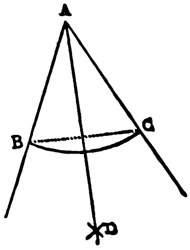


PROP. VI. PROBLEM.

To divide a given angle into two equal parts.

Let BAC be the given angle. With A as a centre, and any radius, describe the arc BC; and from B and C as centres, with any radius longer than the half of the chord BC, describe arcs, intersecting at D; draw the line AD, and it will divide the angle BAC into two equal

Fig. 47.



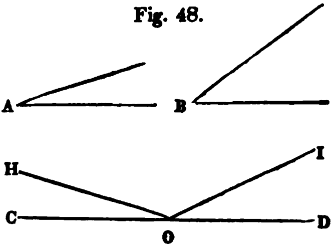
parts; because the points A and D being each equally distant from the extremities of the line BC, AD bisects the chord and arc BC (B. II. Prop. 6); and consequently the angle BAC (B. II. Prop. 4).

PROP. VII. PROBLEM.

Two angles of a triangle being given, to find the third.

Let A, B, be the given angles. Draw a line CD, and at any point O, make the angles COH, DOI, respectively equal to A, B (by Prob. 4), and the angle HOI is the required angle. For the required angle, being the third angle of a triangle, is equal to the difference between two right angles and the sum of A and B (B. I. Prop. 12, Cor. 1); but HIO is the difference between two right angles and the sum of A and B. (B. I. Prop. 1, Cor. 1); hence HIO is the angle required.

Fig. 48.

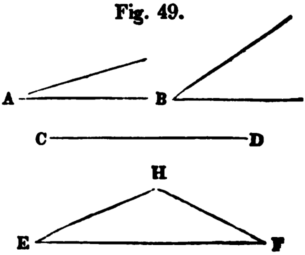


PROP. VIII. PROBLEM.

A side and two angles of a triangle being given, to describe the triangle.

Let A, B, be the given angles, CD the given side. Now these angles must either be both adjacent to the side, or one must be adjacent and the other opposite.

Fig. 49.



In the first case, take a line EF equal to the given side, and make (by Prop. 4) the angle FEH=A, EFH=B;

the lines EH, FH, will cut each other at H, and EHF will be the triangle required.

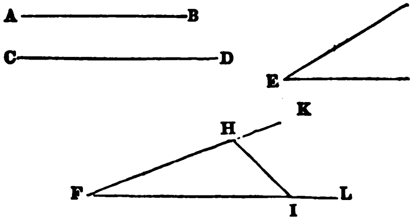
In the second case, where one of the angles is opposite the given side, find the third angle (by Prob. 7); then, having the two adjacent angles, proceed as above.

PROP. IX. PROBLEM.

Two sides and the included angle of a triangle being given, to construct the triangle.

Let AB, CD, be the sides, E the angle. On any line FL make the angle LFK=E; take FI=CD, and FH =AB. Draw HI, and the triangle FIH is the one required.

Fig. 50.



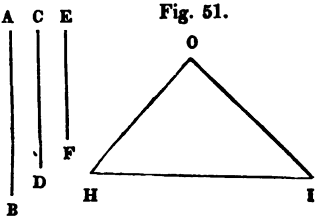
PROP. X. PROBLEM.

The three sides of a triangle being given, to construct the triangle.

Let AB, CD, EF, be the given sides.

Draw HI equal to AB; from H as a centre, with a radius equal to CD, describe an arc; from I as a centre, with a radius equal to EF, describe another arc intersecting the former in O; draw HO, IO, and HIO is the triangle required.

Fig. 51.



PROP. XI. PROBLEM.

Two sides of a triangle and an angle opposite on them being given, to construct the triangle.

There are two cases : 1*st*, when the given angle is r or obtuse ; 2*d*, when it is acute. We will conside *first* case, and take the angle a right angle. Let AB, CD be the given lines, E the given angle.

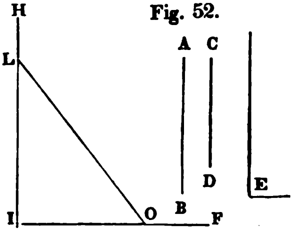


Fig. 52.

Draw the lines IH, IF, at right angles to each other, so as to make the angle I equal to E. On HI take a distance IL e to CD, the shorter line ; and from L as a centre, w radius equal to AB, the longer line, describe an arc ting IF in O ; draw LO, and OLI will be the tria required. In case the given angle is obtuse, the struction is precisely similar, and the longest sid above, must be opposite the given angle (B. I. Prop.

We will now consider the *second* of the above c where the given angle is acute. In this case, if the opposite the given angle is longer than the other given side, the construction is precisely the same as above : we will therefore attend to the case where the side opposite the acute angle is the shorter of the two given sides. Let AB, CD, be the given sides, and E the given angle. Draw

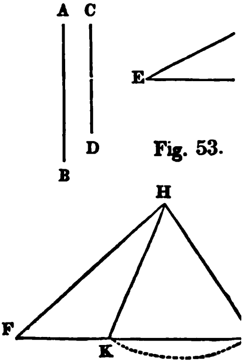


Fig. 53.

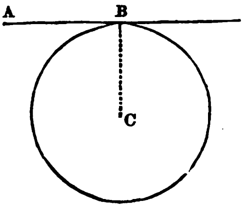
any line FL, and make the angle LFH = E. On the line FH, take a distance FH equal to AB, the longer side; and with H as a centre, and CD, the shorter side, as radius, describe an arc which will cut FL in two points K, L; draw the lines HK, HL, and there will be *two* triangles FHK, FHL, which will satisfy the conditions of the question; that is, two triangles, each of which contains the two given sides and the given included angle.

Scholium. If CD is not of sufficient length to reach the line FL, there will be no triangle which can be formed of the given parts; if it touches FL without cutting it, there will be a right-angled triangle formed, and but one solution.

PROP. XII. PROBLEM.

Through a given point, to draw a tangent to a given circle.

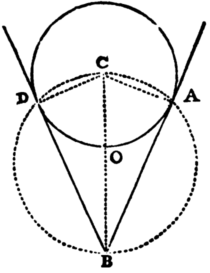
Fig. 54.



1st. Let the given point B be *on* the circumference. Draw the radius from C to B; and at the point B draw AB, perpendicular to BC, and AB will be the tangent required (B. II. Prop. 9).

2d. Let the given point B be without the circle.

Fig. 55.



Join the centre of the given circle and the given point by the line CB; bisect this line in the point O; and with O as a centre, and OC as radius, describe a circumference; it will meet the given circumference in the two points A, D; draw BA, BD, and either of these will be the tangent required; for draw the radii CA, CD, and the angles CAB, CDB, will be right angles, because each is subtended by a semi-circumference (B. II. Prop. 9, and B. II. Prop. 11, Cor. 3).

Cor. The right-angled triangles BDC, BAC, having CB common and CD=CA, are equal (B. I. Prop. 20), and hence BD=BA; therefore *two tangents, drawn to the same circle from a point without, are equal.*

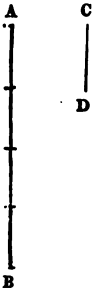
PROP. XIII. PROBLEM.

To find the common measure of two given straight lines.

Let the given lines be AB, CD.

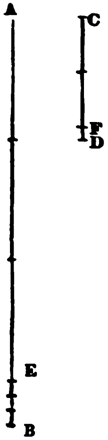
First, suppose CD is contained in AB an exact number of times, that is, without a remainder; say, for example, four times; then is CD itself the common measure sought; for it is contained in itself once, and in AB, four times. Moreover, any part of CD, as one half, one third, one fourth, &c., is also a common measure of these lines; since it obviously will be contained an exact number of times in CD, and consequently in AB.

Fig. 56.



Secondly, suppose that CD is not contained an exact number of times in AB; but that, after it has been contained in it a certain number of times, (say three, for example,) there is left a remainder EB. Now apply this remainder EB to CD, suppose it is contained twice with a remainder FD; apply this last remainder FD, to EB, and suppose it is contained four times without a remainder. Now since FD is contained exactly in EB, it must be contained exactly in CF, which is the double of EB, and consequently in CD; and since it is contained in CD it must be contained in AE, which is three times CD, and consequently in AB; hence FD, being contained exactly in CD and AB, is the common measure of these lines. The given lines are supposed, in the above cases, to have a common measure, and are called *commensurable*.

Fig. 57.



It may happen, indeed, that however far we carry the above operation, we shall never find a remainder which is contained an exact number of times in the two lines: in this case the lines are called *incommensurable*; but it is obvious in this case that the smaller the measuring line is, the smaller the remainder will be; and we may easily suppose the measure taken so small, that the remainder may be neglected without error, and then the lines, to all practical purposes, would be commensurable, that is, have for a common measure this supposed extremely small line.

BOOK IV.

DEFINITIONS.

1. THE *ratio* of two numbers, is the quotient which arises from dividing one by the other: for example, the ratio of 6 to 2 is 3. Ratio is sometimes expressed by writing the numbers in the form of a fraction; as $\frac{6}{2}$, or more frequently thus, 6 : 2, which signifies 6 divided by 2.

Since Geometry relates to *magnitudes*, and not to *numbers*, it is necessary to show, that the ratio of magnitudes of the same kind may be expressed by numbers. Let us take, for example, two *lines*; now to find the ratio between them, or, what is the same thing, the number of times that one is contained in the other, we may suppose them both divided into equal parts, (for example, inches,) and let us say that one contains *sixty-two* inches, and the other *thirty-seven* inches; now the number of times that one of the lines contains the other, will evidently be the same as the number of times which sixty-two contains thirty-seven; that is, the ratio of the lines would be expressed by $\frac{62}{37}$, or 62 divided by 37. Hence the same rules which relate to the ratio of numbers will apply to the ratio of magnitudes.

2. Four numbers are in *proportion*, when the ratio, or quotient, of the first and second, is the same as the ratio of the third and fourth: for example, 12, 6, 8, 4 are in pro-

portion, because 12 divided by 6, is equal to 8 divided by 4.

To indicate a proportion, numbers are written thus; 12 : 6 : : 8 : 4, which is read, 12 is to 6, as 8 is to 4.

3. The numbers which make a proportion are called the *terms* of the proportion; the first and last terms are called the *extremes*, the second and third the *means*.

4. The first and third terms are called *antecedents*, the second and fourth *consequents*.

5. Three numbers are in proportion when the ratio of the first and second is equal to that of the second and third: for example, 12 : 6 : : 6 : 3; in this case the middle term is said to be a *mean proportional* to the other two.

6. When two numbers are multiplied by the same number the results are called *equimultiples* of the two numbers: as, if 8 and 6 are both multiplied by 4, then 32 and 24 are equimultiples of 8 and 6.

7. If we have several numbers multiplied by the same number, as 6.4, 8.4, 10.4, we may express the sum of all these products thus, (6+8+10).4. (Explanation of Signs, 6).

Prop. 1. *When four numbers are in proportion, the product of the means is equal to the product of the extremes;*

as, 6 : 4 : : 15 : 10, hence 10.6=15.4.

Prop. 2. *If the product of two numbers is equal to the product of two other numbers, the four may be arranged in a proportion;*

as, 8.3=12.2, hence 8 : 12 : : 2 : 3.

Scholium. Remark that the test of a proportion, in every case, is, that the product of the extremes is equal to that of the means; hence, when we have four quantities to arrange in a proportion, the terms may be placed

BOOK V.

DEFINITIONS.

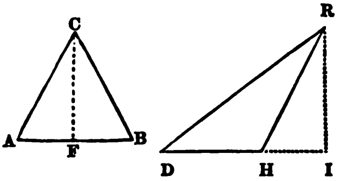
1. *Similar figures* are those, which have the angles of the one respectively equal to the angles of the other, and the sides containing the equal angles proportional.

2. The *homologous* sides of similar figures are those, which are each lying between two angles respectively equal.

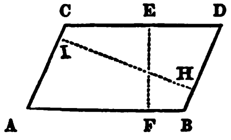
3. In different circles, *similar arcs*, *sectors*, and *segments* are those whose arcs subtend equal angles at the centre.

4. The altitude of a triangle is the perpendicular let fall from the vertex of an angle upon the base, or upon the base produced.

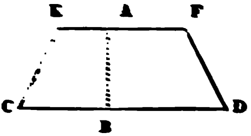
In the triangle ABC, if AB is taken as the base, CF is the altitude. In the triangle DHR, the base is DH, and the altitude RI.



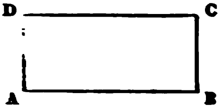
5. The altitude of a parallelogram is the perpendicular between two opposite sides taken as bases. Thus, EF, or IH, is the altitude of the parallelogram ABDC, according as we assume the sides AB, CD, or AC, BD, as the bases.



6. The altitude of a trapezoid is the perpendicular drawn between its two parallel sides.



7. A rectangle is said to be *contained* by its adjacent sides. Thus the rectangle ABCD is contained by the sides AB, BC.



8. A *linear unit* is a line adopted as the measure of length; as an inch, a foot, a yard, &c.

9. A *superficial unit* is a surface, adopted as the measure of surface; as a square inch, square yard, acre, &c.

10. The *area* of a figure is the measure of its surface.

11. Figures with equal areas are called *equivalent*.

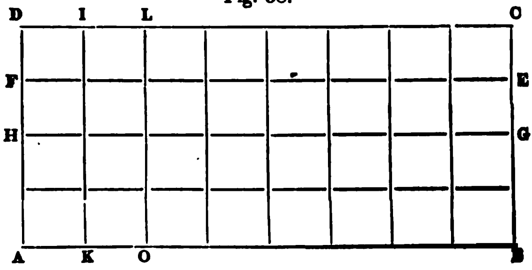
12. The square of a line, as AB, is expressed thus, AB^2, read, "AB square."

13. The rectangle of two lines, as AB, CD, is written AB.CD, read, "AB multiplied into CD," or, shortly, "AB into CD."

PROP. I. THEOREM.

The area of a rectangle is obtained by multiplying the number of linear units in the base, by the number of the same linear units in the altitude.

Fig. 58.



Let the base of the rectangle ABCD contain a certain number of linear units, say eight, it is evident that the rectangle will contain as many times eight superficial units, as there are linear units in the altitude. For from the points of division K, O, &c. draw lines parallel to AD ; and from the points of division E, G, &c. draw lines parallel to AB, and there will be as many squares formed, as the product of the number of linear units in the base, by the number of the same units in the altitude. In the case before us, the area contains thirty-two times the superficial unit, which is a square whose side is the assumed linear unit.

Scholium. Whenever the base and altitude are commensurable, we may take for the linear unit the greatest common measure of them ; and the superficial unit will be a square having the linear unit for a side. For example, if the base of a rectangle was $4\frac{1}{2}$ inches, and its altitude $3\frac{1}{4}$ inches, then $\frac{1}{4}$ inch would be a common measure ; hence the base would contain 18, and the altitude 13 linear units, each $\frac{1}{4}$ inch in length. The superficial unit would be a square, whose side was $\frac{1}{4}$ inch ; and hence would be $\frac{1}{16}$ of a square inch ; therefore the supposed rectangle would contain 18.13 or 234 sixteenths of a square inch, or $\frac{234}{16}$ square inches $=14\frac{10}{16}$ square inches.

If the base and altitude are incommensurable, we may still take the common measure so small (agreeably to Prop. 13 of B. IV.) that the part neglected will be practically unimportant.

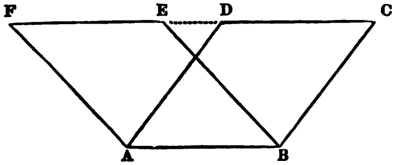
(See Appendix, Prob. II.)

PROP. II. THEOREM.

Parallelograms which have equal bases and equal altitudes are equivalent.

Let ABCD, ABEF, be the parallelograms. The bases, being equal, will, on being applied to each other, coincide in AB; and, since they have the same altitude, their upper bases will be in the same straight line FC.

Fig. 59.



Now DC and EF are each equal to AB (B. I. Prop. 24); hence they are equal to each other (B. I. Ax. 1); add to each ED, and we have EC=FD (B. I. Ax. 4). Again, FA=EB and AD=BC (B. I. Prop. 24); hence the triangles AFD, BEC, having the three sides AF, AD, DF, in one, respectively equal to the three sides EB, BC, EC, of the other, are equal (B. I. Prop. 22). Now if from the whole figure ABCF we take away the triangle AFD, there is left the parallelogram ABCD; and if from the whole figure we take away the equal triangle BCE, there is left the parallelogram ABEF; hence these parallelograms are equivalent (B. I. Ax. 4).

Cor. 1. Every parallelogram is equivalent to a rectangle which has an equal base and equal altitude.

Cor. 2. Hence the area of a parallelogram is equal to the product of its base by its altitude (Prop. 1).*

Cor. 3. Hence parallelograms of equal altitudes, are in proportion to each other as their bases; and parallelograms of equal bases, are to each other as their altitudes.

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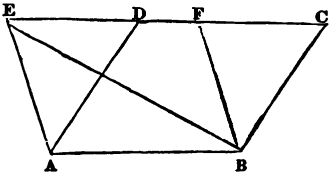
* See Appendix, Problem II.

PROP. III. THEOREM.

Every triangle is half a parallelogram of the same base and altitude.

Let ABE be a triangle, and ABCD a parallelogram having the same base AB and the same altitude, that is, the vertex of the triangle and the upper base of the parallelogram being in the same straight line. From the point B draw BF parallel to the side AE; we shall have a parallelogram ABFE, which is equivalent to the parallelogram ABCD (Prop. 2). But the triangle ABE is one half of the parallelogram ABFE (B. I. Prop. 24, Cor.); hence it is half of the equal parallelogram ABCD.

Fig. 60.



Cor. 1. All triangles having equal altitudes and equal bases are equivalent; because they are halves of equivalent parallelograms.

Cor. 2. Hence triangles of equal altitudes are to each other as their bases, and conversely.

Cor. 3. A triangle is half a rectangle of the same base and altitude.

Cor. 4. Hence the area of a triangle is half the product of its base and altitude.*

PROP. IV. THEOREM.

If a straight line be divided into two parts, the square described on the whole line will be equivalent to the sum of the squares on the two parts, together with double the rectangle contained by those parts.

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* See Appendix, Problem II.

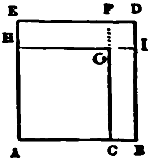


Fig. 61.

Let AB be the line; and let it be divided into the two parts AC, CB, then we have to prove that $AB^2=AC^2+CB^2+2AC.CB$.

Construct the square ABDE; take AH=AC; draw HI parallel to AB, and CF parallel to AE. The square ABDE is made up of four parts; *the first* ACOH is the square described on AC, since we made AH=AC: *the second* OIDF is the square described on OI, or its equal BC; for, since we made AB=AE, and AC=AH, the difference AB—AC must be equal to AE—AH, which gives CB=HE; but OI is equal to CB, and ID equal to HE, since the lines are parallel; therefore OIDF is a square described on BC: *the third* is the rectangle HOFE which is contained by the lines HO, EH, respectively equal to AC, CB; therefore the rectangle HQFE is equal to a rectangle contained by AC, CB: *the fourth* is the rectangle CBIO, which, because CO=AC, is also equal to a rectangle contained by AC, CB; hence the large square is equivalent to the two small squares together with the two rectangles.

Cor. If the line AB were divided into equal parts, the rectangles would become squares, and the square on the whole line would be equivalent to four times the square on half the line.

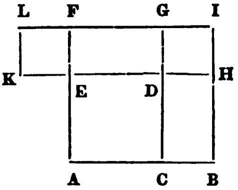
PROP. V. THEOREM.

The square described on the difference of two lines is equivalent to the sum of the squares of the two lines, diminished by twice the rectangle contained by the lines.

Let AB, BC, be the two lines, AC is their difference;

then we have to prove that $AC^2=AB^2+BC^2-2AB.BC$.

Fig. 62.



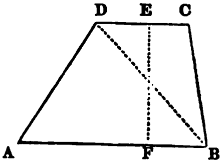
Describe the square ABIF; take AE=AC; draw CG parallel to BI, and HE parallel to AB; continue HE till EK=EF, and complete the square EFLK, which is evidently equal to the square on CB; hence the whole figure is composed of the square on AB, together with the square on CB. If we take away from the whole figure the two rectangles CBIG, KDGL, there will remain the square on AC: now each of these rectangles is equal to a rectangle contained by AB, CB; hence the square on AC is equivalent to the sum of the squares on AB, CB, diminished by twice the rectangle contained by AB, CB.

PROP. VI. THEOREM.

The area of a trapezoid is equal to its altitude multiplied by half the sum of its parallel bases.

Let ABCD be the trapezoid, and EF its altitude; we have to prove that the area of the figure is equal to $\frac{EF(AB+DC)}{2}$.

Fig. 63.



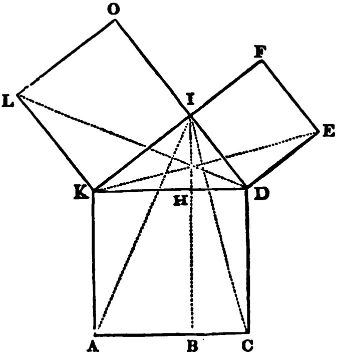
Draw the diagonal DB. EF is the altitude of the triangle ABD with AB taken as the base; and is also the altitude of DBC with DC taken as a base. Now the area of the trapezoid is equal to the sum of the areas of the two triangles; but the area of ADB is equal to $\frac{EF.AB}{2}$ (Prop. 3, Cor. 4), and the area of BDC is equal to $\frac{EF.DC}{2}$; hence the sum of their areas, or the area of the trapezoid, is equal to $\frac{EF(AB+DC)}{2}$.

(See Appendix, Problem IV.)

PROP. VII. THEOREM.

The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.

Fig. 64.



Let the triangle be KDI, right angled at I. Describe squares on KD, KI, DI; then we have to prove that the square KDCA is equivalent to the two squares KIOL, IDEF, taken together. Draw the line IH, perpendicular to the hypotenuse, and produce it to B. Draw IA, DL.

The angles LKI, DKA, are equal, being right angles; add IKD to each, and LKD=IKA (B. I. Ax. 4). Again, LK=KI, being both sides of the same square; and, for the same reason, DK=KA; hence, in the triangles LKD, IKA, we have the sides LK, KD, and the included angle LKD of the one, equal respectively to IK, KA and the included angle IKA of the other, therefore these triangles are equal (B. I. Prop. 4). Taking LK as the base of the triangle LKD, its altitude will be a line drawn from D perpendicular to LK produced, which line will be parallel and equal to KI (B. I

Prop. 10, Cor. 2, and Prop. 24); hence the triangle LKD is half the square LKIO, because they have the same base LK, and the same altitude KI (Prop. 3, Cor. 3): the altitude of the triangle IKA, to the base KA, is a line from I perpendicular to AK produced, therefore equal to HK; consequently the triangle IKA is half the rectangle KHBA, because they have the same base KA, and the same altitude HK.

Now we have shown that the triangles LKD, IKA, are equal, and that one is half the square KIOL, and the other half the rectangle KHBA; therefore, since the halves of the square and rectangle are equivalent, the wholes must be equivalent (B. I. Ax. 3), and the rectangle KHBA is equivalent to the square on KI. In precisely a similar manner it may be shown that the rectangle HDCB, is equivalent to the square on ID; hence the sum of these rectangles, or the square on KD, is equivalent to the squares on KI, DI, taken together.

Cor. 1. The square on either side of a right-angled triangle, is equivalent to the square on the hypotenuse, diminished by the square on the other side.

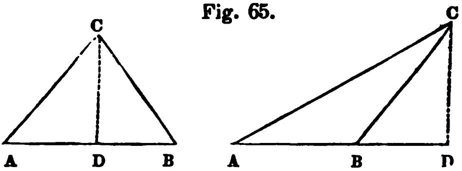
Cor. 2. If the two sides of a right-angled triangle are equal, the square on the hypotenuse is double the square on one side; hence the square on the diagonal of a square is double the square on a side.

(See Appendix, Problem I.)

PROP. VIII. THEOREM.

In any triangle, the square of a side opposite the acute angle is equal to the sum of the squares of the other sides, diminished by twice the rectangle contained by the base and the distance from the acute angle to the foot of the perpendicular.

Fig. 65.



The perpendicular may fall within, or without the base; the demonstration applies to both cases. Let A be the acute angle, CB the opposite side.

DB is equal to the difference between AB and AD; hence $DB^2=AB^2+AD^2-2AB.AD$, (Prop. 5); add to each of these equals CD^2 and the results will be equal (B. I. Ax. 4); but $DB^2+CD^2=CB^2$ (Prop. 7), and $AD^2+CD^2=AC^2$; therefore when CD^2 is added to each of the above equals, we shall have $CB^2=AB^2+AC^2-2AB.AD$.

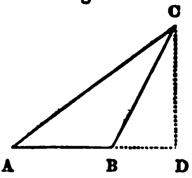
PROP. IX. THEOREM.

In any triangle having an obtuse angle, the square of the side opposite thereto is equal to the sum of the squares of the two other sides, increased by twice the rectangle of the base and distance of the obtuse angle from the foot of the perpendicular.

Let ABC be the triangle.

AD is equal to the sum of AB, BD; hence $AD^2=AB^2+BD^2+2AB.BD$ (Prop. 4); add CD^2 to each of these equals; and, since $AD^2+CD^2=AC^2$, and $BD^2+CD^2=BC$, we shall have $AC^2=AB^2+BC^2+2AB.BD$.

Fig. 66.

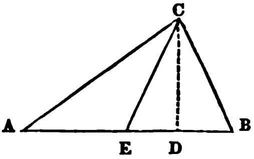


PROP. X. THEOREM.

In any triangle, if a straight line be drawn from the vertex to the middle of the base, twice the square of this line, together with twice the square of half the base, is equivalent to the sum of the squares of the other two sides of the triangle.

Let the triangle be ACB, and CE the line drawn to the middle of the base.

Fig. 67.



Draw the perpendicular CD; then in the triangle CEB, CB being opposite the acute angle CEB, we have (by Prop. 8) $CB^2 = EB^2 + CE^2 - 2EB.ED$; and in the obtuse-angled triangle ACE, $AC^2 = AE^2 + CE^2 + 2AE.ED$ (Prop. 9); hence, adding these equals together, and remarking that $AE = EB$, we get

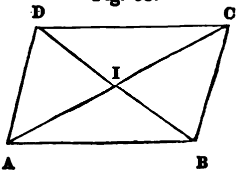
$$AC^2 + CB^2 = 2AE^2 + 2CE^2 - 2AE.ED + 2AE.ED;$$

now, since $2AE^2 + 2CE^2$ is both increased and diminished by $2AE.ED$, its magnitude will not be altered; and, therefore, we may omit $2AE.ED$; then we shall have $AC^2 + CB^2 = 2AE^2 + 2CE^2$.

PROP. XI. THEOREM.

In every parallelogram, the sum of the squares of all the sides is equivalent to the sum of the squares of the diagonals.

Fig. 68.



The diagonals AC, BD, bisect each other (B. I. Prop. 27); hence, by the last Proposition,

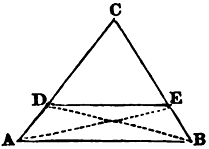
$AD^2+DC^2=2AI^2+2DI^2$, and
$AB^2+BC^2=2AI^2+2BI^2$; adding these equals together, $AD^2+DC^2+AB^2+BC^2=4AI^2+4DI^2$; but $4AI^2$ is the square of AC (Prop. 4, Cor.), and $4DI^2$ is the square of DB ; hence

$$AD^2+DC^2+AB^2+BC^2=AC^2+DB^2.$$

PROP. XII. THEOREM.

If a line be drawn parallel to one of the sides of a triangle, it will cut the other sides proportionally.

Fig. 69.



Let DE be drawn parallel to the side AB of the triangle ABC; then we have to prove that

AD : DC : : BE : EC.

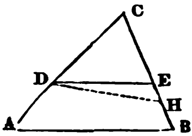
Draw AE, BD ; then the triangles AED, BDE, having the same base DE, and the same altitude, (since their vertices are in the same line AB parallel to the base,) are equivalent (Prop. 3, Cor. 1). Again, the two triangles AED, DEC, having the same altitude, (because their vertices are at the same point E,) are to each other as their bases (Prop. 3, Cor. 2); that is, AED : DEC : : AD : DC ; and for a similar reason we have BDE : DEC : : BE : EC ; but we have shown that AED= BDE ; hence in these two proportions, an antecedent and consequent being the same in each, the remaining antecedent and consequent are proportional (B. IV. Prop. 4, Cor. 1) ; therefore, AD : DC : : AE : EC.

PROP. XIII. THEOREM.

(Converse of Prop. XII.)

If a straight line be drawn cutting two sides of a triangle proportionally, it will be parallel to the third side.

Fig. 70.



Let DE be so drawn that AD : DC : : BE : EC ; then we have to prove that DE is parallel to AB.

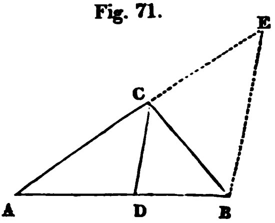
If it is not parallel, draw DH parallel to AB ; then (Prop. 12) AD : DC : : BH : HC ; but, by the hypothesis, AD : DC : : BE : EC ; hence (B. IV. Prop. 4, Cor. 1) BH : BE : : HC : EC ; but this proportion cannot be true, because BH being shorter than BE, while HC is longer than EC, BE cannot be contained in BH as often as EC is contained in HC (B. IV. Def. 2); now this absurdity arose by regarding DH as parallel to AB ; hence DH cannot be parallel to AB ; and as this can be shown of every other line drawn through D except DE, it follows that DE must be parallel to AB.

PROP. XIV. THEOREM.

If a line be drawn bisecting any angle of a triangle, it will divide the opposite side into two parts which are proportional to the adjacent sides.

Let the angle ACB, of the triangle ACB, be bisected by the line CD ; then we have to prove that AD : DB : : AC : CB.

Fig. 71.



Through B draw the line BE, parallel to CD, till it meets AC produced. In the triangle ABE, since CD is parallel to the side BE, we have AD : DB : : AC : CE (Prop. 12). The line AE cutting the parallels CD, BE makes the opposite exterior and interior angles ACD, CEB, equal (B. I. Prop. 9, Cor. 1); and CB cutting the same parallels makes the alternate angles DCB, CBE, equal (B. I. Prop. 9); but by hypothesis ACD, DCB, are equal; hence CEB, CBE, must be equal; therefore the triangle CEB, having the angles at its base equal, is isosceles (B. I. Prop. 7), and CE is equal to CB. Now substitute, in the above proportion, for CE its equal CB, and it will become AD : DB : : AC : CB.

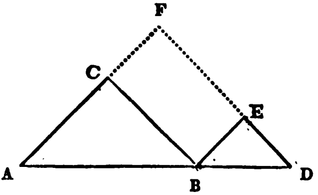
PROP. XV. THEOREM.

If two triangles are equiangular, their homologous sides are proportional.

Let the triangles ABC, BDE be equiangular, having the angle CAB=EBD, ACB=BED, ABC=BDE; then we have to prove that the sides between these equal angles are proportional.

Let the triangles be so placed that the homologous sides AB, BD, are in one straight line, and produce the sides AC, DE, till they meet in F. Now since the opposite exterior and interior angles CAB, EBD, are equal, the lines AF, BE, are parallel (B. I. Prop. 10, Cor. 1), and since ABC=BDE, the lines CB, DF, are parallel (for the same reason as above); hence the figure CFBE is a parallelogram (B. I. Def. 17), and the opposite sides are equal (B. 1. Prop. 24); that is, CF=BE, BC=EF. Now in the triangle ADF, because BE is parallel to AF, we have (Prop. 12) BD : AB : : DE : EF; and in the same tri-

Fig. 72.



angle having BC parallel to DF, we get AB : BD : : AC : CF. In these proportions put for EF and CF, their equals CB and BE, and they become

BD : AB : : DE : CB, (1)

AB : BD : : AC : BE, or by inversion (IV. Prop. 5)

BD : AB : : BE : AC ; now in this last proportion and the one marked (1) an antecedent and consequent are the same in both ; hence the other antecedents and consequents are proportional (B. IV. Prop. 4, Cor. 1); hence DE : BE : : CB : AC.

Cor. Two equiangular triangles are similar (Def. 1).

PROP. XVI. THEOREM.

(Converse of Prop. XIV.)

Two triangles which have their sides proportional, are equiangular and similar.

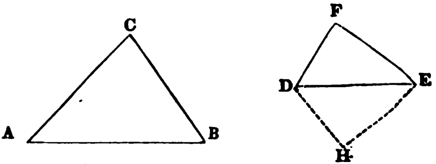
If, in the triangles ABC, DEF,

AB : AC : : DE : DF, and

AB : BC : : DE : FE,

then we have to prove that these triangles are equiangular.

Fig. 73.



At the point D, draw DH so as to make the angle EDH equal to BAC ; and at the point E, draw the line EH, so as to make the angle DEH equal to ABC ; then the third angle H will be equal to C (B. I. Prop. 12, Cor. 2); hence (by the last Proposition) we have AB : AC : : DE : DH ; but by the hypothesis AB : AC : : DE : DF ; hence (B. IV. Prop. 4, Cor. 2) DE : DH : : DE . DF ;

hence (B. IV. Prop. 8, Cor.) DH=DF. Again (by the last proposition) we have AB : BC : : DE : EH, and by the hypothesis AB : BC : : DE : FE ; hence, as before, EH=FE ; therefore the triangles DEF, DEH, having their three sides equal, each to each, are themselves equal (B. I. Prop. 22); but we made the triangle DEH equiangular with ABC ; hence its equal DEF must also be equiangular with ABC ; and therefore (by the Corollary to the last Proposition) similar.

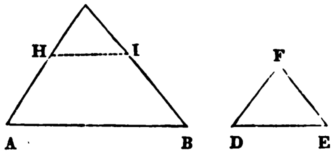
PROP. XVII. THEOREM.

Two triangles, which have an angle of the one equal to an angle of the other, and the sides about the equal angles proportional, are similar.

In the triangles ABC, DEF, let the angles, C, F, be equal, and AC : CB : : DF : FE, then we have to prove that these triangles are similar.

Take CH=DF, and CI=FE, draw HI; then are the triangles CHI, FDE, equal (B. I. Prop. 4).

Fig. 74.



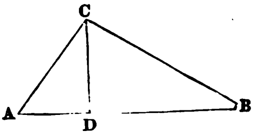
By the hypothesis we have AC : CB : : DF : FE ; putting for DF, FE, their equals HC, CI, this proportion becomes AC : CB : : CH : CI ; hence (Prop. 13) HI is parallel to AB, and the angles CHI, CIH, are respectively equal to CAB, CBA, (B. I. Prop. 10, Cor. 1) now since the triangle CHI is equiangular with ABC, its equal DEF must also be equiangular with ABC, and hence similar.

PROP. XVIII. THEOREM.

The perpendicular from the vertex of the right angle to the hypotenuse of a right-angled triangle, divides it into two triangles similar to each other, and similar to the whole triangle.

Let ABC be the triangle, right-angled at C, draw the perpendicular CD; then we have to prove that the triangles ACD, BCD, are each similar to ABC, and consequently similar to each other.

Fig. 75.



The triangles ACD, ABC, have the angle A common, and have each a right angle; they are therefore equiangular (B. I. Prop. 12, Cor. 2) and similar. Again, the triangles BCD, ABC, have the angle B common, and have each a right angle; hence they are similar; therefore the triangles ACD, BCD, being both similar to ABC, are similar to each other.

Cor. On account of the similarity of the triangles ACD, BCD, we have AD : CD : : CD : DB, that is, the perpendicular is a mean proportional (B. IV. Def. 5) between the parts into which it divides the hypotenuse.

PROP. XIX. THEOREM.

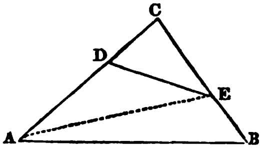
Triangles, having an angle in the one equal to an angle in the other, are to each other as the rectangle of the sides which contain the equal angle.

Let the triangles ABC, CDE, have the equal angle C; then we have to prove that

ABC : DEC : : AC.CB : CD.CE.

Draw AE. The triangles CAE, CED, having a common vertex E, have the same altitude, and are therefore to each other as their bases (Prop. 3, Cor. 2), that is, CAE : DEC : : AC : DC ; and because the triangles ABC, CAE, have a common vertex at A, we shall have ABC : CAE : : CB : CE. Multiply the corresponding terms of these proportions (B. IV. Prop. 10), and omitting the term CAE, which would be common to the antecedent and consequent (B. IV. Prop. 8), we shall have ABC : DEC : : AC.CB : DC.CE.

Fig. 76.

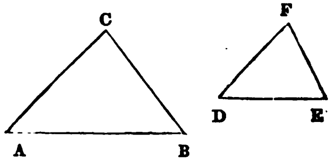


PROP. XX. THEOREM.

Two similar triangles are to each other as the squares of their homologous sides.

Let ABC, DEF, be the similar triangles, having the angles A, B, C, respectively equal to D, E, F ; we have to prove that the triangle ABC : the triangle DEF : : $AC^2 : DF^2$.

Fig. 77.



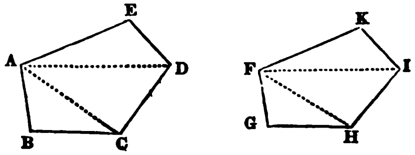
On account of the equal angles A, D, according to the last Proposition, (1) ABC : DEF : : AC.AB : DF.DE ; also, because the triangles are similar, AB : DE : : AC : DF (Prop. 15); hence (B. IV. Prop. 1) AB.DF=DE.AC ; multiplying each of these equals by AC.DF we get $AB.AC.DF^2 = DE.DF.AC^2$; now arranging this in a proportion (B. IV. Prop. 2), AB.AC : DE.DF : : $AC^2 : DF^2$; comparing this with the proportion above marked (1) (B. IV. Prop. 4, Cor. 1), we get ABC : : DEF : : $AC^2 : DF^2$.

PROP. XXI. THEOREM.

Two similar polygons are composed of the same number of triangles, which are respectively similar, and like situated.

Let ABCDE, FGHIK, be the similar polygons. From any angle, as A in the first, draw the diagonals AD, AC; and from the homologous angle F, in the other, draw the diagonals FI, FH. The polygons being similar, the angles B, G, must be equal, and the sides containing them proportional (Def. 1); that is, AB : BC : : FG : GH; hence the triangles ABC, FGH, having each an equal angle contained between proportional sides, are similar (Prop. 17), and the angles BCA, GHF, equal; take away these equals from the equal angles BCD, GHI, there remains ACD=FHI. Since the triangles ABC, FGH, are similar, we have AC : BC : : FH : GH, and since the polygons are similar, DC : BC : : IH : GH; hence (B. IV. Prop. 4, Cor. 2) we obtain AC : DC : : FH : IH; therefore the triangles ACD, FHI, having an equal angle in each, and the sides about the equal angle proportional, are similar (Prop. 17).

Fig. 78.



In the same manner it may be shown that the remaining triangles are similar.

PROP. XXII. THEOREM.

The perimeters (B. I. Def. 11) *of similar polygons are to each other as the homologous sides; and the areas are as the squares of those sides.*

Since the polygons are similar (last figure), we have this proportion, AB : FG : : BC : GH : : CD : HI: : ED : IK : : AE : FK.

Now the sum of all the antecedents, which, makes up the perimeter of the first polygon, is to the sum of all the consequents, the perimeter of the other polygon, as either antecedent is to its consequent (B. IV. Prop. 6); that is, the perimeter of ABCDE : the perimeter of FGHIK : : AB : FG.

Again, since the triangles ABC, FGH,are similar, we have (by Prop. 20) ABC : FGH : : AC^2 : FH^2; and since ACD, FHI, are similar, ACD : FHI : : AC^2 : FH^2; hence (B. IV. Prop. 4, Cor. 2) ABC : FGH: : ACD : FHI; by the same mode of reasoning we should find ACD : FHI : : ADE : FIK; writing these last two proportions as a continued proportion,

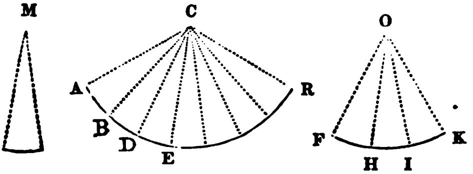
ABC : FGH : : ACD : FHI : : ADE : FIK; hence, (B. IV. Prop. 6), ABC+ACD+ADE : FGH+FHI+FIK : : ABC : FGH : : AB^2 : FG^2; that is, the polygon ABCDE : the polygon FGHIK : : AB^2 : FG^2.

PROP. XXIII. THEOREM.

In equal circles, two angles at the centre are to each other as the arcs intercepted between their sides.

Suppose the angle M to be the common measure of the angles ACR, FOK, and that it is contained seven times in ACR, and three times in FOK; then will the angle ACR be to the angle FOK in the proportion of

Fig. 79.



seven to three; that is, ACR : FOK : : 7 : 3; now the angles ACB, BCD, DCE, &c., FOH, HOI, IOK, being each equal to the angle M, are equal to each other; and therefore the arcs AB, BD, DE, &c., FH, HI, IK, are all equal (B. II. Prop. 4); hence the arc AR is divided into seven and the arc FK into four equal parts; so that the arc AR : the arc FK : : 7 : 4; hence, since the arcs bear the same proportion to each other that the angles do, we have the angle ACR : angle FOK : : arc AR : arc FK.

It is evident that the above demonstration would be true, however small M might be, or how many soever times it might be contained in the other angles, provided it was an exact number of times, that is, that there was no remainder. But the angles may be incommensurable, that is, have no measure which shall be contained in them both without leaving a remainder in one or the other; in this case, however, we may conceive the measure to be taken so small that the remainder, which would be still smaller, may be neglected, and the angles regarded without error as commensurable. Thus in every case it may be proved that in equal circles, angles at the centre are as their arcs.

Cor. Hence an arc may always be taken as the measure of the angle at the centre which it subtends. The circumference of the circle is divided into 360 degrees, each degree into 60 minutes, and each minute into sixty seconds; and we speak of an angle of so many degrees, &c., meaning an angle at the centre subtended by an arc of that number of degrees, &c.

BOOK VI.

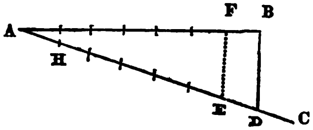
PROBLEMS.

PROP. I. PROBLEM.

To divide a given straight line into any number of equal parts.

Let AB be the given line, and suppose it is required to divide it in seven equal parts. From A, draw the line AC in any direction; take AH of any length, and apply it seven times upon AC; join the last point of division D and the extremity B of the given line, then through E draw EF parallel to BD, and BF will be one seventh of AB, so that, applying BF upon the line AB, it will divide it into seven equal parts. For, since EF is parallel to BD, the lines AB, AD, are cut proportionally in F and E (B. V. Prop. 12); but ED is one seventh of AD, hence BF is one seventh of AB.

Fig. 80.

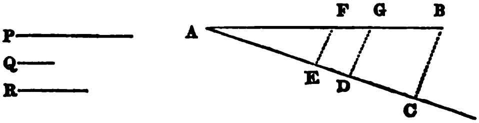


PROP. II. PROBLEM.

To divide a given line into parts proportional to given lines.

Let it be proposed to divide the given line AB into parts proportional to the given lines P, Q, R. Through A, draw AC in any direction, make AE=P, ED=Q, DC=R; join the point C with the extremity B of the given line, and through D, E, draw DG, EF, parallel to BC; then will the line AB be divided into parts AF, FG, GB, proportional to the given lines P, Q, R. For, by reason of the parallels DG, EF, the parts AF, FG, GB, are proportional to AE, ED, DC, and these were made equal respectively to P, Q, R.

Fig. 81.

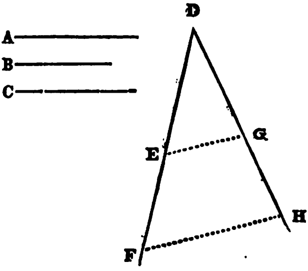


PROP. III. PROBLEM.

To find a fourth proportional to the given lines A, B, C.

Draw the indefinite lines DF, DH, making any angle with each other. Take DE =A, EF=B, DG=C; join the points E, G, and from F, draw FH parallel to EG; then GH is the fourth proportional required, because DE : EF : : DG : GH; but DE, EF, DG, are respectively equal to A, B, C.

Fig. 82.



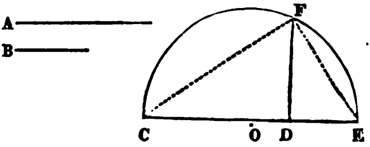
Cor. If DG is made equal to EF, that is, equal to B, GH will be a *third proportional* to the two given lines A, B.

PROP. IV. PROBLEM.

To find a mean proportional between two given lines A, B.

Draw an indefinite line CE, and make CD=A, DE=B, find O the middle of the line CE, and with OC as radius describe a semicircle; from the point D draw a perpendicular to CE meeting the circumference in F; then is DF the mean proportional required. For draw FC, FE, and the triangle formed will be right angled at F (B. II. Prop. 11, Cor. 3); hence DF is a mean proportional between CD, DE, which were made equal respectively to A, B, (Prop. 18, Cor.)

Fig. 83.

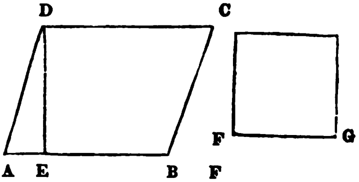


PROP. V. PROBLEM.

To describe a square that shall be equivalent to a given parallelogram.

Let ABCD be the given parallelogram, AB the base, DE its altitude. Find a mean proportional FG between AB and DE; then the square erected on FG will be equivalent to the parallelogram. For by construction AB : FG : : FG : DE, therefore (B. IV. Prop. 1) AB.DE=FG^2; but AB.DE is the area of the parallelogram (B. V. Prop. 2, Cor. 2) and FG^2 the area of the square; therefore they are equivalent.

Fig. 84.

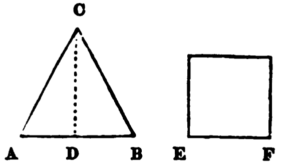


PROP. VI. PROBLEM.

To describe a square that shall be equivalent to a given triangle.

Let ABC be the given triangle, AB its base, CD its

Fig. 85.



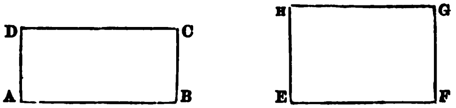
altitude. Find a mean proportional between AB and the half of CD; let EF be that mean, and the square erected on EF will be equivalent to the triangle. For by the construction we have AB : EF : : EF : $\frac{1}{2}$CD; hence AB.$\frac{1}{2}$CD$=$EF^2; but AB.$\frac{1}{2}$CD is the area of the triangle (B. V. Prop. 3, Cor. 4), and EF^2 the area of the square, consequently the square and triangle are equivalent.

PROP. VII. PROBLEM.

Upon a given line to describe a rectangle that shall be equivalent to a given rectangle.

Let EF be the given line, ABCD the given rectangle. Find a fourth proportional to the three lines EF, AB, BC; let FG be that proportional; a rectangle constructed with the lines EF, FG, will be equivalent to the rectangle ABCD. For, since EF : AB : : BC : FG, we have EF.FG$=$AB.BC; but the first of these equals is the measure of the rectangle EFGH, and the second of the rectangle ABCD (B. V. Prop. 1); hence these rectangles are equivalent.

Fig. 86.

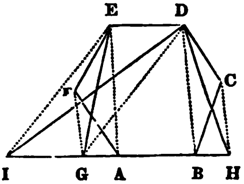


PROP. VIII. PROBLEM.

To find a triangle that shall be equivalent to a given polygon.

Let ABCDEF be the given polygon. From the point A, draw the diagonal AE, and from F, draw FG parallel to AE, meeting the base produced in G; draw EG; the polygon ABCDEF will be equivalent to the polygon GBCDE, which has one side less than the original polygon.

Fig. 87.



For the triangles AEF, AEG, have the same base AE, and the same altitude, since their vertices F, G, are in the same line FG parallel to the base; therefore these triangles are equivalent (B. V. Prop. 3, Cor. 1). Now if to the triangle AEF we add the figure ABCDE, there will result the polygon ABCDEF; and if to the triangle AEG we add the same figure, there will result the polygon GBCDE, hence these polygons are equivalent (B. I. Ax. 4).

Now, having reduced our original polygon of six sides, to its equivalent one GBCDE, which has but five sides, proceed as before; that is, draw GD, and from E draw EI parallel to GD, cutting the base produced in I; then join ID, and because the triangles GDE, GDI, are equivalent, if we add each of them to GBCD, we shall find IBCD equivalent to GBCDE, therefore to ABCDEF. Proceed in like manner with the angle C and we shall find the triangle IHD equivalent to the polygon ABCDEF.

Scholium. We have already seen that every triangle may be changed into an equivalent square (Prob. 6.), and as we can find a triangle equivalent to any polygon, we can always find a square equivalent to a given polygon; this is called *squaring* the polygon, or finding the *quadrature* of it.

BOOK VII.

DEFINITIONS.

1. POLYGONS of five sides are called *pentagons*; those of six sides, *hexagons*; of seven, *heptagons*; of eight, *octagons*, &c.

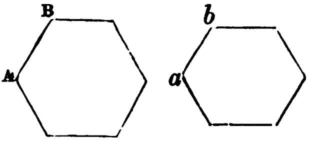
2. Polygons which are at once equilateral and equiangular, are called *regular polygons*.

PROP. I. THEOREM.

Two regular polygons of the same number of sides, are similar.

Let the figures in the margin be two regular polygons, having the same number of sides. The sum of all the angles in each figure is equal to twice as many right angles as the figure has sides, less four right angles (B. I. Prop. 13), and as the number of sides is the same in each figure, the sum of all the angles in each must be the same. Now since all the angles are equal (Def. 2), any one angle is found by dividing the sum of all the angles by the number of angles; hence it follows that the angles of the two polygons are equal, each to each.

Fig. 88.



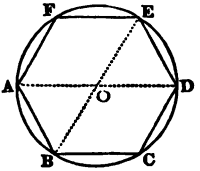
Again, since the sides in each polygon are all equal, it follows that AB : BC : : *ab* : *bc*; and so for all the sides; hence these polygons, having the angles of the one respectively equal to the angles of the other, and the sides including the equal angles proportional, are similar (B. VI. Def. 1).

PROP. II. THEOREM.

The side of a regular hexagon, inscribed in a circle, is equal to the radius of that circle.

Let ABCDEF be a regular hexagon. Now since the arcs subtended by equal chords are equal, the three arcs AB, BC, CD, are together equal to the three arcs AF, FE, ED; hence AD is a diameter: for similar reasons BE is also a diameter.

Fig. 89.



Now the inscribed angle ADE, standing on the arc AFE, is equal to BED, standing on the equal arc BCD; and both are equal to the angle at the centre EOD, standing on the arc ED, half of AFE, or BCD (B. II. Prop. 2, and Cor. 2); hence the triangle EOD, having all its angles equal, is equilateral (B. I. Prop. 7, Cor.), and ED is equal to EO.

Scholium 1. Hence in order to inscribe a regular hexagon in a circle, apply the radius six times round the circumference. By joining the alternate points by the lines AC, CE, EA, an inscribed equilateral triangle will be formed.

Scholium 2. If we bisect the arcs AB, BC, &c., and from the points A, B, C, &c., draw lines to the points of bisection, it is evident that a regular inscribed figure will be formed, having twelve sides; and that the surface of this figure will be more nearly equal to that of the circle, than is the surface of the hexagon.

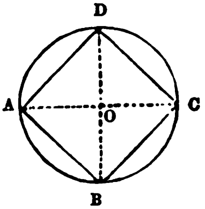
Again, if we bisect the arcs of our twelve-sided figure, one of twenty-four sides will be formed, which will approach still nearer to the circle than the last; therefore we may conclude generally, that the greater number of sides, which a regular inscribed figure has, the more nearly will its surface approach to that of the circle.

PROP. III. PROBLEM.

To inscribe a square in a given circle.

Draw the diameters AC, BD cutting each other at right angles; join their extremities A, B, C, D; the inscribed figure formed will be a square. For the angles AOD, DOC, AOB, BOC, being equal, the chords AD, DC, CB, BA, are equal (B. II. Prop. 3, Cor. 1); and the angles ADC, DCB, CBA, BAD, being each subtended by a semi-circumference, are right angles (B. II. Prop. 2, Cor. 3).

Fig. 90.

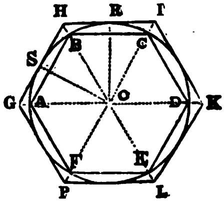


PROP. IV. PROBLEM.

A regular inscribed polygon being given, to circumscribe a similar one about the same circle.

Let ABCDEF be the regular inscribed polygon. Bisect the arcs AB, BC, &c., in the points S, R, &c., and at these points draw GH, HI, &c., tangents to the circle; these lines will be

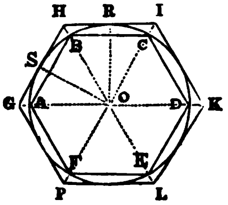
Fig. 91.



respectively parallel to AB, BC, &c. (because GH, AB, &c. are perpendicular to OS, &c.)

Now these tangents, by their intersections, will form a circumscribed polygon GHIKLP, which we have to prove will be similar to the inscribed one.

Fig. 91.



Draw from H, the point where the tangents HG, HI, intersect, the line HO; we shall show that it will pass through B. The arcs BRC, BSA, being equal, their halves BR, BS, must be equal, and consequently the point B is the middle of the arc SR; now the right-angled triangles SOH, HOR, having the sides OS, OR, equal, and OH common, are equal (B. I. Prop. 20), and consequently the angle SOH = HOR; hence the line from H to O bisects the arc SBR, and must therefore pass through the point B.

Since the lines GH, HI, are respectively parallel to AB, BC, the angles GHI, ABC, are equal (B. I. Prop. 8); in the same manner all the angles of the circumscribing polygon are equal to the angles of the inscribed one, and since these latter are all equal to each other, the former are so also.

The triangles GOH, AOB, being equiangular, we have (B. V. Prop. 15)

OH : OB : : HG : BA ;

and the triangles OHI, OBC, being equiangular,

OH : OB : : HI : BC ;

hence (B. IV. Prop. 4, Cor. 1)

HG : BA : : HI : BC ;

but BA is equal to BC, hence (B. IV. Prop. 8, Cor.)

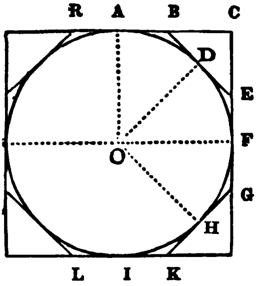
HG=HI; in the same manner we can show HI=IK=KL, &c.; hence, since we have proved that the circumscribing polygon is equiangular and equilateral, it is, according to the definition, regular; and having the same number of sides with the inscribed polygon, is similar to it (Prop. 1).

PROP. V. PROBLEM.

A regular circumscribed polygon being given, it is required to circumscribe the circle by another regular polygon of double the number of sides of the given one.

Circumscribe the circle with a square, and bisect the arcs AF, FI, &c., at the points D, H, &c.; through these points draw tangents BE, GK, &c., till they meet the sides of the square, then the figure LKGEBR, &c., will be the required polygon. In order to prove this, draw OA, OD, OH, and suppose the quadrilateral ODEF placed upon the quadrilateral OFGH, so that OF shall remain as it is: since the angles DOF, FOH, are equal, (each being measured by half the equal arcs FA, FI,) the line OD will take the direction OH, and the point D will fall on H; and since the angles OFE, OFG, are right angles, the line FE will take the direction FG; and since the angles ODE, OHG, are also right angles, the line DE will take the direction HG, and consequently the point E will fall on G, and the quadrilaterals will exactly coincide; hence DE=HG, FE=FG, and the angle DEF=FGH. By applying in a similar manner the quadrilaterals ODEF, ODBA, it may be shown

Fig. 92.



that DE=DB, EF=BA, and the angle DEF=DBA. But since the tangents ED, EF, are equal (B. III. Prop. 12, Cor.), it follows that EB, which is twice ED, is equal to EG, which is twice EF.

In a similar manner it may be shown that all the sides and all the angles of the octagon are equal; therefore it is regular. This construction is applicable to any other polygons.

Scholium. It is evident that the surface of the octagon approaches nearer that of the circle, than the surface of the square does; and that, if we circumscribe the circle with a figure of sixteen sides, its surface would approach still nearer to that of the circle; hence *the greater the number of sides of the regular circumscribed polygon, the nearer its surface approaches to that of the circle.*

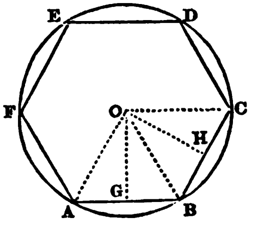
PROP. VI. THEOREM.

The area of a regular polygon is equal to its perimeter multiplied by half the line drawn from the centre of the circumscribed circle perpendicular to any side.

Let ABCDEF be the regular polygon, OG, OH, lines drawn from the centre of the circumscribed circle perpendicular to the sides AB, AF.

The chords AB, AF, FE, &c. being equal, it follows that the perpendiculars OG, OH, &c. are also equal (B. II. Prop. 8). Now the area of each of the triangles AOB, AOF, &c. is equal respectively to AB, AF, &c. multiplied by the half of OG, OH, &c.; therefore the area of the whole polygon is equal to its perimeter multi-

Fig. 93.



plied by half of the perpendicular from the centre of the circumscribed circle to any side.

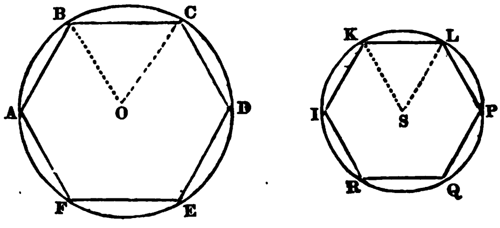
Cor. The greater the number of sides of the inscribed polygon, the nearer its surface approaches to that of the circle (Prop. 3, Sch. 2); and we may suppose the number of sides to be so great, that the surface of the polygon will coincide with that of the circle, its perimeter become equal to the circumference of the circle, and the lines OG, OH, become radii; hence this rule: *The area of a circle is equal to its circumference multiplied by half its radius.**

PROP. VII. THEOREM.

The perimeters of two regular polygons of the same number of sides, are to each other as the radii of the circumscribed circles: and their areas are to each other, as the squares of those radii.

Let ABCDEF, IKLPQR, be two regular polygons of the same number of sides; they will therefore be simi-

Fig. 94.



lar (Prop. 1), and the angle OBC, which is half the angle ABC, is equal to SKL, which is half IKL; and, for a similar reason, the angles OCB, SLK, are equal; therefore the angle O is equal to the angle S (B. I. Prop. 12, Cor.2); and the two triangles OBC, SKL, are similar (B. V. Prop. 15, Cor.); and we have BC : KL : : BO : KS ; but the perimeters are to each other as BC : KL (B. V. Prop. 22), hence the perimeters are to each other as BO : KS.

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* See Appendix, Prob. VI.

Again, since BC : KL : : BO : KS, we have BC^2 : KL^2 : BO^2 : KS^2 (B. IV. Prop. 7); and since the surfaces of the polygons are to each other as BC^2 : KL^2 (B. V. Prop. 22), they will also be to each other as BO^2 : KS^2.

PROP. VIII. THEOREM.

The circumferences of circles are to each other as their radii ; and their surfaces as the squares of those radii.

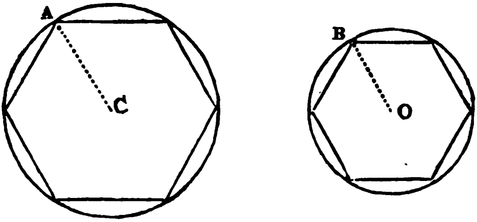
Let us designate the circumference of the circle whose radius is CA, by *circ* CA ; and its area, by *area* CA ; and use similar contractions with respect to the circle whose radius is OB ; it is then to be shown that

circ CA : *circ* OB : : CA : OB,

and that

area CA : *area* OB : : CA^2 : OB^2.

Fig. 95.



Inscribe within the circles two regular polygons of the same number of sides ; then whatever be the number of their sides, their perimeters will be to each other as the radii CA, OB, (Prop. 7). Now the arcs subtending the sides of the polygons may be continually bisected, until the perimeters of the polygons become equal to the circumferences of the circles (Prop. 6, Cor.); hence we shall have

$$\textit{circ}\ AC : \textit{circ}\ BO :: AC : BO.$$

Again, the areas of the polygons are to each other as $AC^2 : BO^2$ (Prop. 7); and, when the number of sides of the polygons be so increased that the areas of the polygons are equal to the areas of the circles, we shall have

$$\textit{area}\ AC : \textit{area}\ BO :: AC^2 : BO^2.$$

Scholium 1. The circumference of a circle whose diameter is unity, has been determined to be, very nearly, 3.1416. The circumferences of circles being to each other as their radii, they will also be to each other as their diameters (B. IV. Prop. 9); hence to determine the circumference of a circle whose radius is CA, say

$$1 : 2CA :: 3.1416 : \textit{circ}\ CA;$$

that is, the diameter of one circle, is to the diameter of another, as the circumference of the first, is to the circumference of the second; from the above proportion we get,

$$\textit{circ}\ CA = 2CA \times 3.1416;$$

that is,* *To find the circumference of a circle whose radius is given, multiply twice the radius by* 3.1416.

Scholium 2. Since the area of a circle is equal to its circumference multiplied by half its radius (Prop. 6, Cor.), in order to find the area of a circle whose radius is known, we must compute its circumference by the rule given above, and multiply that circumference by half its radius. In case of a circle whose radius is CA, the circumference is equal to $2CA \cdot 3.1416$; multiply this by $\frac{1}{2}CA$, and the area becomes equal to $CA^2 \times 3.1416$; hence this rule: *To find the area of a circle whose radius is known, multiply the square of its radius by* 3.1416.†

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* See Appendix, Prob. V.

† See Appendix, Prob. VI.

APPENDIX.

APPLICATIONS.

PROBLEM I. RIGHT-ANGLED TRIANGLES.

1. In a right-angled triangle having the base 40 feet, the perpendicular 9 feet, to find the hypotenuse.

	Square feet.
(B. V. Prop. VII.) the square of the base,	1600
the square of the perpendicular,	81
the square of the hypotenuse,	1681

Ans. therefore the hypotenuse 41 feet.

2. Having the base 3 feet, the perpendicular 4 feet, to find the hypotenuse. *Ans.* 5 feet.

3. Having the base 8 yards, and the perpendicular 6 yards, to find the hypotenuse. *Ans.* 10 yards.

4. Having the hypotenuse 61 feet, the perpendicular 11 feet, to find the base.

(B. V. Prop. VII, Cor. 1)

	Square feet.
the square of the hypotenuse,	3721
do. do. perpendicular,	121
do. do. base,	3600

Ans. therefore the base 60 feet.

5. Having the hypotenuse 15 feet, the base 9 feet, to find the perpendicular. *Ans.* 12 feet.

6. Having the hypotenuse 25 feet, the base 7 feet, to find the perpendicular. *Ans.* 24 feet.

7. Having the hypotenuse 85 feet, the perpendicular 84 feet, to find the base. *Ans.* 13 feet.

PROBLEM II. (B. V. PROPS. I. AND II.)

Area of square, rectangle, or parallelogram.

1. What is the area of a rectangle, whose length is 11 feet and breadth 13 feet? 13 multiplied by 11 is= *Ans.* 143 sq. ft.

2. What is the area of a square, whose side is 6½ feet? 6½ reduced to decimals is 6.5; and 6.5 multiplied by 6.5 is *Ans.* 42.25 sq. ft.

3. What is the area of a square, whose side is 204.3 feet? *Ans.* 41738.49 sq. ft.

4. What is the contents, in square yards, of a rectangle whose base is 66.6 feet, and altitude 33.3 feet?

Reduce the feet to yards, and multiply. *Ans.* 246.42.

5. Find the area of a rectangular board, whose length is 12½ feet, and breadth 9 inches, or ¾ foot.

Reduce the fractions to decimals. *Ans.* 9.375 sq. ft.

6. Find the number of square feet of painting in a parallelogram whose base is 37 feet, and whose altitude is 5¼ feet. *Ans.* 194.25.

7. Find the number of square yards in a floor, whose length is 5.33 yds. and breadth 4.25 yds. *Ans.* 22.6525.

PROBLEM III. (B. V. PROP. III. COR. IV.)

1. Find the area of a triangle, whose base is 625 feet, and altitude 520 feet. *Ans.* 162500 sq. ft.

2. Find the number of square yards in a triangle, whose base is 40 feet and altitude 30 feet.

Find the number of square feet, and then divide by 9, the number of square feet in a square yard. *Ans.* 66⅔.

3. Find the number of square yards in a triangle, whose base is 49 feet and altitude $25\frac{1}{4}$ feet. *Ans.* 68.7361.

PROBLEM IV. (B. V. PROP. VI.)

In a trapezoid, the parallel sides 750 feet and 1225 feet, and the perpendicular distance between them 1540 feet, what is the area? *Ans.* 1520750 sq. ft.

2. How many square feet are contained in a plank, whose length is $12\frac{1}{2}$ feet, breadth at the greater end 15 inches, and at the less end 11 inches? *Ans.* 13.541.

3. How many square yards in a trapezoid, whose parallel sides are 240 feet, 320 feet, and altitude 66 feet? *Ans.* 2053.33.

PROBLEM V. (B. VII. PROP. VIII. SCH. I.)

To find the circumference of a circle when the radius is known.

1. What is the circumference of a circle whose radius is $12\frac{1}{2}$? *Ans.* 78.54.

2. If the diameter of the earth is 7921 miles, what is the circumference? *Ans.* 24884.6136.

3. What is the diameter of a circle whose circumference is 11652.1944?

Divide the circumference by 3.1416. *Ans.* 37.09.

4. What is the diameter of a circle whose circumference is 6850? *Ans.* 2180.41.

PROBLEM VI. (B. V. III. PROP. VI. COR., AND PROP. VIII. SCH. II.)

To find the area of a circle.

1. Find the area of a circle whose diameter is 10, and circumference 31.416. *Ans.* 78.54.

2. Find the area of a circle whose diameter is 7, and circumference 21.9912. *Ans.* 38.4846.

3. What is the area of a circle whose radius is 6 feet? *Ans.* 113.0976.

4. How many square yards in a circle whose diameter is 3½ feet? *Ans.* 1.06901.

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